

A Note on Invariantly Finitely L -Presented Groups

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Abstract

In the first part of this note, we introduce Tietze transformations for L -presentations. These transformations enable us to generalize Tietze's theorem for finitely presented groups to invariantly finitely L -presented groups. Moreover, they allow us to prove that 'being invariantly finitely L -presented' is an abstract property of a group which does not depend on the generating set.

In the second part of this note, we consider finitely generated normal subgroups of finitely presented groups. Benli proved that a finitely generated normal subgroup of a finitely presented group is invariantly finitely L -presented whenever its quotient is infinite cyclic. We generalize this result to the case where the finitely presented group splits over its finitely generated subgroup and to the case where the quotient is abelian with torsion-free rank at most two.

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1 Introduction

Finite L -presentations are possibly infinite group presentations with finitely many generators whose relations (up to finitely many exceptions) are obtained by iteratively applying finitely many substitutions to a finite set of relations; see [1] or Section 2 for a definition. Various infinitely presented groups can be described by a finite L -presentation. For example, the Grigorchuk group [6] and the Gupta-Sidki group [9] are finitely L -presented [17, 19, 1, 2]. An L -presentation is *invariant* if the substitutions, which generate the relations, induce endomorphisms of the group. In fact, invariant finite L -presentations are finite presentations in the universe of *groups with operators* [15, 18] in the sense that the operator domain of the group generates the possibly infinitely many relations out of a finite set of relations. The finite L -presentation for the Grigorchuk group in [17] is an example of an invariant finite L -presentation [7].

Finite L -presentations allow computer algorithms to be applied in the investigation of the groups they define. For instance, they allow one to compute the lower central series quotients [2], to compute the Dwyer quotients of the group's Schur multiplier [10], to develop a coset enumerator for finite index subgroups [11], and even the Reidemeister-Schreier theorem for finitely presented groups generalizes to finitely L -presented groups [13]. For a survey on the application of computers in the investigation of finitely L -presented groups, we refer to [12].

In the first part of this note, we introduce Tietze transformations for L -presentations. These transformations allow us to generalize Tietze's theorem for finitely presented groups [20] to invariantly finitely L -presented groups:

Theorem A *Two invariant finite L -presentations define isomorphic groups if and only if it is possible to pass from one L -presentation to the other by a finite sequence of transformations.*

If a group admits a finite presentation with respect to one generating set, then so it does with respect to any other finite generating set [5, Chapter V]. This result for finitely presented groups also generalizes to invariant finite L -presentations:

Theorem B (Bartholdi [1]) *Being invariantly finitely L -presented is an abstract property of a group which does not depend on the generating set.*

Our proof of Theorem B fills a gap in the proof of [1, Proposition 2.2] because the transformations in the latter proof are not sufficient; see Section 4 below.

In the second part of this note, in Section 5, we consider finitely generated normal subgroups of finitely presented groups. By Higman's embedding theorem, every finitely generated group embeds into a finitely presented group if and only if it is recursively presented [14]. Since every finite L -presentation is recursive, finitely L -presented groups therefore embed into finitely presented groups. As indicated in [4], we prove that every group which admits an invariant finite L -presentation, where each substitution induces an automorphism of the group, embeds as a *normal* subgroup into a finitely presented group. On the other hand, the Reidemeister-Schreier theorem for finitely L -presented groups in [13] shows that every normal subgroup of a finitely presented group admits an invariant L -presentation where each substitution induces an automorphism of the group; the obtained L -presentation is finite if and only if the normal subgroup has finite index.

Finitely generated *normal* subgroups of finitely presented groups with infinite index were considered in [4]: It was proved that a finitely generated normal subgroup of a finitely presented group is invariantly finitely L -presented if its quotient is infinite cyclic. Moreover, in [4, Remark (2)], Benli asked for a generalization of his latter result and he posed the following problem:

Is it true that a finitely generated group embeds as a normal subgroup into a finitely presented group if and only if it admits an invariant finite L -presentation where each substitution induces an automorphism of the group?

We generalize Benli's constructions from [4] in order to prove the following

Theorem C *Every finitely generated normal subgroup of a finitely presented group is invariantly finitely L -presented if the group splits over its subgroup.*

Since G splits over its subgroup $H \trianglelefteq G$ if G/H is a free group, Benli's result in [4] is a consequence of Theorem C. Moreover, our generalizations of the constructions from [4] allows us to prove

Theorem D *Every finitely generated normal subgroup of a finitely presented group is invariantly finitely L -presented whenever the quotient is abelian with torsion-free rank at most two.*

Our constructions do not generalize further; see Remark 2.1.

2 Preliminaries

In this section, we recall the notion of an invariant finite L -presentation as introduced in [1]. An L -presentation is a group presentation of the form

$$\langle \mathcal{X} \mid \mathcal{Q} \cup \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle, \quad (1)$$

where \mathcal{X} is an alphabet, \mathcal{Q} and \mathcal{R} are subsets of the free group $F = F(\mathcal{X})$ over the alphabet \mathcal{X} , and $\Phi^* \subseteq \text{End}(F)$ denotes the monoid of endomorphisms that is generated by Φ . If the *generators* \mathcal{X} , the *fixed relations* \mathcal{Q} , the *substitutions* Φ , and the *iterated relations* \mathcal{R} have finite cardinality, the L -presentation in Eq. (1) is a *finite L -presentation*. We also write $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ for the L -presentation in Eq. (1) and $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ for the group it defines.

A group which admits a finite L -presentation is *finitely L -presented*. An L -presentation of the form $\langle \mathcal{X} \mid \emptyset \mid \Phi \mid \mathcal{R} \rangle$ is *ascending* and an L -presentation $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is called *invariant* (and the group it defines is *invariantly L -presented*), if each substitution $\varphi \in \Phi$ induces an endomorphism of the group; i.e., if the normal subgroup $\langle \mathcal{Q} \cup \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle^F \trianglelefteq F$ is φ -invariant. Each ascending L -presentation is invariant and each invariant L -presentation $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ admits an ascending L -presentation $\langle \mathcal{X} \mid \emptyset \mid \Phi \mid \mathcal{Q} \cup \mathcal{R} \rangle$ which defines the same group; see Proposition 3.4. Even though invariant and ascending L -presentations are essentially the same, we like to distinguish between these two objects. The finite L -presentation in [17] for the group constructed by Grigorchuk [6] is not ascending but it is easy to see that it is an invariant L -presentation; see, for instance, [7, Corollary 4].

Remark 2.1 *There are finite L -presentations that are not invariant.*

Proof. The free product $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle \{a, b\} \mid \{a^2, b^2\} \rangle$ is finitely L -presented by $\langle \{a, b\} \mid \{a^2\} \mid \{\sigma\} \mid \{b^2\} \rangle$ where σ is induced by the map $a \mapsto ab$ and $b \mapsto b^2$. If this L -presentation were invariant, the ascending L -presentation $\langle \{a, b\} \mid \emptyset \mid \{\sigma\} \mid \{a^2, b^2\} \rangle$ would also define $\mathbb{Z}_2 * \mathbb{Z}_2$; see Proposition 3.4. In this case $(a^2)^\sigma = abab$ is a relation in the group and, since $a^2 = b^2 = 1$ holds, the generators a and b commute. Therefore the ascending L -presentation defines a quotient of the 2-elementary abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$. In fact, it defines a finite group. Thus $\langle \{a, b\} \mid \emptyset \mid \{\sigma\} \mid \{a^2, b^2\} \rangle$ is not a finite L -presentation for $\mathbb{Z}_2 * \mathbb{Z}_2$ and hence $\langle \{a, b\} \mid \{a^2\} \mid \{\sigma^2\} \mid \{b^2\} \rangle$ is not an invariant L -presentation. \square

Note that this latter proof from [13] provides a ‘method’ to prove that a finite L -presentation $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is invariant; namely, if the ascending L -presentation $\langle \mathcal{X} \mid \emptyset \mid \Phi \mid \mathcal{R} \cup \mathcal{Q} \rangle$ defines a group which is isomorphic to the first. In general, we are not aware of a method which allows us to decide whether or not a finite L -presentation is invariant — even if we assume that the L -presented group has a solvable word problem.

Invariant finite L -presentations are ‘natural’ generalizations of finite presentations because every finitely presented group $\langle \mathcal{X} \mid \mathcal{R} \rangle$ is invariantly finitely L -presented by $\langle \mathcal{X} \mid \emptyset \mid \emptyset \mid \mathcal{R} \rangle$. However, invariant finite L -presentations have been used to describe various examples of self-similar groups that are not finitely presented [17, 3]. For instance, the group \mathfrak{G} constructed by Grigorchuk in [6] is not finitely presented [8] but it is invariantly finitely L -presented, see also [7]:

Theorem 2.2 (Lysënok [17]) *The Grigorchuk group is invariantly finitely L -presented by $\langle \{a, b, c, d\} \mid \{a^2, b^2, c^2, d^2, bcd\} \mid \{\sigma\} \mid \{(ad)^4, (adacac)^4\} \rangle$ where σ denotes the endomorphism of the free group over $\{a, b, c, d\}$ that is induced by the map $a \mapsto aca$, $b \mapsto d$, $c \mapsto b$, and $d \mapsto c$.*

It is easy to see (and it follows with our Tietze transformations below) that the group \mathfrak{G} is also invariantly finitely L -presented by

$$\mathfrak{G} \cong \langle \{a, c, d\} \mid \{a^2, c^2, d^2, (cd)^2\} \mid \{\tilde{\sigma}\} \mid \{(ad)^4, (adacac)^4\} \rangle, \quad (2)$$

where $\tilde{\sigma}$ is induced by the map $a \mapsto aca$, $c \mapsto cd$, and $d \mapsto c$. Further examples of invariantly finitely L -presented groups arise, for instance, as certain wreath-products: In contrast to [1], Bartholdi noticed that the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ is

invariantly finitely L -presented by

$$\langle \{a, t\} \mid \emptyset \mid \{\delta\} \mid \{a^2, [a, a^t]\} \rangle,$$

where δ is induced by the map $a \mapsto a^t a$ and $t \mapsto t$. This recent result generalizes to wreath products of the form $H \wr \mathbb{Z}$, where H is a finitely generated abelian group:

Proposition 2.3 *If H is a finitely generated abelian group, the wreath product $H \wr \mathbb{Z}$ is invariantly finitely L -presented.*

Proof. Since H is finitely generated and abelian, it decomposes into a direct product of cyclic groups; i.e., H has the form $\mathbb{Z}_{r_1} \times \cdots \times \mathbb{Z}_{r_n}$ for $r_1, \dots, r_n \in \mathbb{N} \cup \{\infty\}$ where \mathbb{Z}_∞ denotes the infinite cyclic group while \mathbb{Z}_{r_i} denotes the cyclic group of order r_i , otherwise. Then $\langle \mathcal{X} \mid \{[x, y] \mid x, y \in \mathcal{X}\} \cup \{x^{r_x} \mid r_x < \infty\} \rangle$ is a finite presentation for H . The wreath product $H \wr \mathbb{Z}$ admits the presentation

$$H \wr \mathbb{Z} \cong \langle \mathcal{X} \cup \{t\} \mid \{[x, y], x^{r_x}\}_{x, y \in \mathcal{X}, r_x < \infty} \cup \{[x, y^{t^i}]\}_{x, y \in \mathcal{X}, i \in \mathbb{N}_0} \rangle.$$

For each $y \in \mathcal{X}$, define a substitution σ_y which is induced by the map

$$\sigma_y: \begin{cases} y & \mapsto y^t y, \\ x & \mapsto x, \\ t & \mapsto t. \end{cases} \quad \text{for each } x \in \mathcal{X} \setminus \{y\},$$

For $n \in \mathbb{N}$ and $x, y, z \in \mathcal{X}$ with $x \neq y$ and $z \neq y$, we obtain

$$\begin{aligned} [y, x^{t^n}]^{\sigma_y} &= [y^t y, x^{t^n}] = [y, x^{t^{n-1}}]^{ty} \cdot [y, x^{t^n}], \\ [x, y^{t^n}]^{\sigma_y} &= [x, y^{t^{n+1}} y^{t^n}] = [x, y^{t^n}] \cdot [x, y^{t^{n+1}}]^{y^{t^n}}, \\ [x, z^{t^n}]^{\sigma_y} &= [x, z^{t^n}], \\ [y, y^{t^n}]^{\sigma_y} &= [y, y^{t^{n-1}}]^{ty} \cdot [y, y^{t^n}]^{ty^{t^n} y} \cdot [y, y^{t^n}] \cdot [y, y^{t^{n+1}}]^{y^{t^n}}. \end{aligned}$$

This shows that the relations $\{[x, y^{t^i}] \mid x, y \in \mathcal{X}, i \in \mathbb{N}\}$ are consequences of the iterated images $\{[x, y^{t^\delta}] \mid \delta \in \{\sigma_y \mid y \in \mathcal{X}\}^*, x, y \in \mathcal{X}\}$ and vice versa. Moreover, for each relation x^{r_x} of H 's finite presentation, we have that $(x^{r_x})^{\sigma_y} = x^{r_x}$ if $x \neq y$ and $(y^{r_y})^{\sigma_y} = (y^t y)^{r_y} =_{H \wr \mathbb{Z}} (y^{r_y})^t y^{r_y}$, otherwise. Thus these images are relations of the wreath product $H \wr \mathbb{Z}$. In particular, the finite L -presentation

$$\langle \mathcal{X} \cup \{t\} \mid \emptyset \mid \{\sigma_y\}_{y \in \mathcal{X}} \mid \{[x, y^{t^i}]\}_{x, y \in \mathcal{X}, i \in \mathbb{N}} \cup \{x^{r_x}\}_{x \in \mathcal{X}, r_x < \infty} \rangle$$

is an invariant finite L -presentation for the wreath product $H \wr \mathbb{Z}$. \square

Even though invariant finite L -presentations are known for numerous self-similar groups, we are not aware of an invariant finite L -presentation for the Gupta-Sidki group from [9]. Moreover, we are not aware of a finitely L -presented group which is not invariantly finitely L -presented.

3 Tietze Transformations for L -Presentations

In this section, we introduce Tietze transformations for L -presentations. Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ be an L -presented group. Denote by F the free group $F(\mathcal{X})$ over the alphabet \mathcal{X} and let $K = \langle \mathcal{Q} \cup \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle^F$ be the kernel of the free presentation $\pi: F \rightarrow G$. Then $K = \ker \pi$ decomposes into the normal subgroups $Q = \langle \mathcal{Q} \rangle^F$ and $R = \langle \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle^F$ so that $K = RQ = QR$ holds. The group F/R is invariantly L -presented by $\langle \mathcal{X} \mid \emptyset \mid \Phi \mid \mathcal{R} \rangle$. We can add every element of the kernel K as a fixed relation:

Proposition 3.1 *If $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is a (finitely) L -presented group and $\mathcal{S} \subseteq \langle \mathcal{Q} \cup \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle^F$ is a (finite) subset, then $\langle \mathcal{X} \mid \mathcal{Q} \cup \mathcal{S} \mid \Phi \mid \mathcal{R} \rangle$ is a (finite) L -presentation for G .*

Proof. The proof follows with the Tietze transformation that adds consequences \mathcal{S} of G 's relations to the group presentation $\langle \mathcal{X} \mid \mathcal{Q} \cup \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle$. \square

The transformation in Proposition 3.1 is reversible in the sense that we can remove fixed relations \mathcal{S} from an L -presentation $\langle \mathcal{X} \mid \mathcal{Q} \cup \mathcal{S} \mid \Phi \mid \mathcal{R} \rangle$ if and only if $\langle \mathcal{Q} \cup \mathcal{S} \cup \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle^F = \langle \mathcal{Q} \cup \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle^F$ holds. The following transformations are reversible in the same sense.

If an L -presentation is not invariant (cf. Remark 2.1), there exist elements from the kernel K of the free presentation $\pi: F \rightarrow G$ that we cannot add as iterated relations without changing the isomorphism type of the group. However, even for non-invariant L -presentations we have the following

Proposition 3.2 *If $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is a (finitely) L -presented group and $\mathcal{S} \subseteq \langle \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle^F$ is a (finite) subset, then $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \cup \mathcal{S} \rangle$ is a (finite) L -presentation for G .*

Proof. By construction, the normal subgroup $R = \langle \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle^F$ is σ -invariant for each $\sigma \in \Phi^*$. More precisely, for each $r \in R$ and $\sigma \in \Phi^*$, we have $r^\sigma \in R$. Therefore, adding the (possibly infinitely many) relations $\{s^\sigma \mid s \in \mathcal{S}, \sigma \in \Phi^*\}$ to the group presentation $\langle \mathcal{X} \mid \mathcal{Q} \cup \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle$ does not change the isomorphism type of the group. \square

Iterated and fixed relations of an L -presentation are related by the following

Proposition 3.3 *If $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is a (finitely) L -presented group and $\mathcal{S} \subseteq \mathcal{R}$ holds, then $\langle \mathcal{X} \mid \mathcal{Q} \cup \mathcal{S} \mid \Phi \mid (\mathcal{R} \setminus \mathcal{S}) \cup \{r^\psi \mid r \in \mathcal{S}, \psi \in \Phi\} \rangle$ is a (finite) L -presentation for G .*

Proof. The proof follows immediately from

$$\mathcal{Q} \cup \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma = \mathcal{Q} \cup \mathcal{S} \cup \bigcup_{\sigma \in \Phi^*} ((\mathcal{R} \setminus \mathcal{S}) \cup \{r^\psi\}_{r \in \mathcal{S}, \psi \in \Phi})^\sigma;$$

these are the relations of G 's group presentation. \square

The following proposition is a consequence of the definition of an invariant L -presentation:

Proposition 3.4 *If $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is an invariant (finite) L -presentation for the group G and $\mathcal{S} \subseteq \mathcal{Q}$ holds, then $\langle \mathcal{X} \mid \mathcal{Q} \setminus \mathcal{S} \mid \Phi \mid \mathcal{R} \cup \mathcal{S} \rangle$ is a (finite) L -presentation for G .*

Proof. Since G is invariantly L -presented by $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$, each $\sigma \in \Phi$ induces an endomorphism of the group G . Therefore, the images $\{q^\sigma \mid q \in \mathcal{S}, \sigma \in \Phi^*\}$ are relations within G and so $\langle \mathcal{X} \mid (\mathcal{Q} \setminus \mathcal{S}) \cup \bigcup_{\sigma \in \Phi^*} (\mathcal{R} \cup \mathcal{S})^\sigma \rangle$ is a presentation for G . \square

The following proposition allows us to add generators together with fixed relations to an L -presentation:

Proposition 3.5 *Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ be an L -presented group, \mathcal{Z} be an alphabet so that $\mathcal{X} \cap \mathcal{Z} = \emptyset$ holds, and, for each $z \in \mathcal{Z}$, let $w_z \in F(\mathcal{X})$ be given. For each $\sigma \in \Phi$, define an endomorphism of the free group E over the alphabet $\mathcal{X} \cup \mathcal{Z}$ that is induced by the map*

$$\tilde{\sigma}: \begin{cases} x \mapsto x^\sigma, & \text{for each } x \in \mathcal{X}, \\ z \mapsto g_z, & \text{for each } z \in \mathcal{Z}, \end{cases} \quad (3)$$

where g_z are arbitrary elements of the free group E . Then G satisfies that

$$G \cong \langle \mathcal{X} \cup \mathcal{Z} \mid \mathcal{Q} \cup \{z^{-1}w_z\}_{z \in \mathcal{Z}} \mid \{\tilde{\sigma}\}_{\sigma \in \Phi} \mid \mathcal{R} \rangle. \quad (4)$$

If $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is a finite L -presentation and \mathcal{Z} is a finite alphabet, the L -presentation in Eq. (4) is finite.

Proof. Write $H = \langle \mathcal{X} \cup \mathcal{Z} \mid \mathcal{Q} \cup \{z^{-1}w_z \mid z \in \mathcal{Z}\} \mid \{\tilde{\sigma} \mid \sigma \in \Phi\} \mid \mathcal{R} \rangle$ and let F and E be the free groups over \mathcal{X} and $\mathcal{X} \cup \mathcal{Z}$, respectively. To avoid confusion, the elements of G 's presentation are denoted by $\bar{g} \in F$. Then

$$\pi: \begin{cases} x \mapsto \bar{x}, & \text{for each } x \in \mathcal{X}, \\ z \mapsto \bar{w}_z, & \text{for each } z \in \mathcal{Z}, \end{cases}$$

induces a surjective homomorphism $\pi: E \rightarrow F$. By construction, the restriction of the substitution $\tilde{\sigma}$ to the free group F coincides with σ . Thus $(\bigcup_{\sigma \in \Phi} \mathcal{R}^{\tilde{\sigma}})^\pi = \bigcup_{\sigma \in \Phi} \mathcal{R}^\sigma$ and hence, π maps iterated relations of H 's L -presentation to iterated relations of G . Similarly, π maps the fixed relations \mathcal{Q} of H 's L -presentation to fixed relations of G . It remains to consider the relations of the form $z^{-1}w_z$ with $z \in \mathcal{Z}$. However, these relations are mapped trivially by π . This shows that the homomorphism $\pi: E \rightarrow F$ induces a surjective homomorphism $\tilde{\pi}: H \rightarrow G$. On the other hand, identifying the generators of G 's L -presentation with the generators of H induces a surjective homomorphism $\varphi: G \rightarrow H$ with $\varphi\tilde{\pi} = \text{id}_H$ and $\tilde{\pi}\varphi = \text{id}_G$. Hence, the groups G and H are isomorphic. The second assertion is obvious. \square

We can also add the relations $\{z^{-1}w_z \mid z \in \mathcal{Z}\}$ in Proposition 3.5 as iterated relations to the L -presentation if we define the substitutions $\tilde{\sigma}$ as follows:

Proposition 3.6 *Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ be an L -presented group, \mathcal{Z} be an alphabet so that $\mathcal{X} \cap \mathcal{Z} = \emptyset$ holds, and, for each $z \in \mathcal{Z}$, let $w_z \in F(\mathcal{X})$ be given. For each $\sigma \in \Phi$, define an endomorphism of the free group E over the alphabet $\mathcal{X} \cup \mathcal{Z}$ that is induced by the map*

$$\tilde{\sigma}: \begin{cases} x \mapsto x^\sigma, & \text{for each } x \in \mathcal{X}, \\ z \mapsto w_z^\sigma, & \text{for each } z \in \mathcal{Z}. \end{cases} \quad (5)$$

Then G satisfies that

$$G \cong \langle \mathcal{X} \cup \mathcal{Z} \mid \mathcal{Q} \mid \{\tilde{\sigma}\}_{\sigma \in \Phi} \mid \mathcal{R} \cup \{z^{-1}w_z\}_{z \in \mathcal{Z}} \rangle. \quad (6)$$

If $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is a finite L -presentation and \mathcal{Z} is a finite alphabet, the L -presentation in Eq. (6) is finite.

Proof. The substitutions $\tilde{\sigma}$ in Eq. (5) are well-defined because $w_z \in F(\mathcal{X})$ and $\sigma \in \text{End}(F(\mathcal{X}))$ hold. By Proposition 3.3, we have that

$$\begin{aligned} & \langle \mathcal{X} \cup \mathcal{Z} \mid \mathcal{Q} \mid \{\tilde{\sigma}\}_{\sigma \in \Phi} \mid \mathcal{R} \cup \{z^{-1}w_z\}_{z \in \mathcal{Z}} \rangle \\ &= \langle \mathcal{X} \cup \mathcal{Z} \mid \mathcal{Q} \cup \{z^{-1}w_z\}_{z \in \mathcal{Z}} \mid \{\tilde{\sigma}\}_{\sigma \in \Phi} \mid \mathcal{R} \cup \{(z^{-1}w_z)^{\tilde{\sigma}}\}_{z \in \mathcal{Z}, \sigma \in \Phi} \rangle. \end{aligned}$$

By definition of $\tilde{\sigma}$ in Eq. (5), we have $(z^{-1})^{\tilde{\sigma}} = (w_z^\sigma)^{-1}$ and $w_z^{\tilde{\sigma}} = w_z^\sigma$. Thus $(z^{-1}w_z)^{\tilde{\sigma}} = (w_z^\sigma)^{-1}w_z^\sigma = 1$ holds. In particular, adding the relations $\{(z^{-1}w_z)^{\tilde{\sigma}} \mid z \in \mathcal{Z}, \sigma \in \Phi\}$ to a group presentation does not change the isomorphism type of the group. By Proposition 3.5, we have that

$$\begin{aligned} G &= \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle \\ &\cong \langle \mathcal{X} \cup \mathcal{Z} \mid \mathcal{Q} \cup \{z^{-1}w_z\}_{z \in \mathcal{Z}} \mid \{\tilde{\sigma}\}_{\sigma \in \Phi} \mid \mathcal{R} \rangle \\ &= \langle \mathcal{X} \cup \mathcal{Z} \mid \mathcal{Q} \cup \{z^{-1}w_z\}_{z \in \mathcal{Z}} \mid \{\tilde{\sigma}\}_{\sigma \in \Phi} \mid \mathcal{R} \cup \{(z^{-1}w_z)^{\tilde{\sigma}}\}_{z \in \mathcal{Z}, \sigma \in \Phi} \rangle \\ &= \langle \mathcal{X} \cup \mathcal{Z} \mid \mathcal{Q} \mid \{\tilde{\sigma}\}_{\sigma \in \Phi} \mid \mathcal{R} \cup \{z^{-1}w_z\}_{z \in \mathcal{Z}} \rangle; \end{aligned}$$

which proves the first assertion of Proposition 3.6 while the second is obvious. \square

The following proposition allows us to modify the substitutions of an L -presentation:

Proposition 3.7 *If $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is a (finitely) L -presented group and $\Psi \subseteq \Phi$ holds, then $\langle \mathcal{X} \mid \mathcal{Q} \mid (\Phi \setminus \Psi) \cup \{\sigma\psi \mid \psi \in \Psi, \sigma \in \Phi\} \mid \mathcal{R} \cup \bigcup_{\psi \in \Psi} \mathcal{R}^\psi \rangle$ is a (finite) L -presentation for G .*

Proof. The proof follows immediately from

$$\mathcal{Q} \cup \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma = \mathcal{Q} \cup \bigcup_{\sigma \in \widehat{\Phi}^*} \left(\mathcal{R} \cup \bigcup_{\psi \in \Psi} \mathcal{R}^\psi \right)^\sigma$$

where $\widehat{\Phi} = (\Phi \setminus \Psi) \cup \{\sigma\psi \mid \psi \in \Psi, \sigma \in \Phi\}$; these are the relations of G 's group presentation. \square

Since each relation of a group presentation can be replaced by a conjugate, we can modify the substitutions of an L -presentation as follows:

Proposition 3.8 *Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ be a (finitely) L -presented group, $\mathcal{S} \subseteq F$ be a (finite) subset, and let $\Psi \subseteq \Phi$ be given. For each $x \in \mathcal{S}$, denote by δ_x the inner automorphism of the free group $F(\mathcal{X})$ that is induced by conjugation with x . Then*

- $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \cup \{\delta_x \mid x \in \mathcal{S}\} \mid \mathcal{R} \rangle$,
- $\langle \mathcal{X} \mid \mathcal{Q} \mid (\Phi \setminus \Psi) \cup \{\delta_x \sigma \mid x \in \mathcal{S}, \sigma \in \Psi\} \mid \mathcal{R} \rangle$, and
- $\langle \mathcal{X} \mid \mathcal{Q} \mid (\Phi \setminus \Psi) \cup \{\sigma \delta_x \mid x \in \mathcal{S}, \sigma \in \Psi\} \mid \mathcal{R} \rangle$

are (finite) L -presentations for G .

Proof. This follows because each relation of a group presentation can be replaced by a conjugate and we have $\delta_x \sigma = \sigma \delta_{x\sigma}$ for each $\sigma \in \Phi^*$ and $x \in \mathcal{X}$. \square

Recall that the kernel $K = \langle \mathcal{Q} \cup \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle^F$ of the free presentation $\pi: F \rightarrow G$ decomposes into the normal subgroups $Q = \langle \mathcal{Q} \rangle^F$ and $R = \langle \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle^F$ so that $K = QR = RQ$ holds. This decomposition yields the following

Proposition 3.9 *Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ be a (finitely) L -presented group and let $\Psi \subseteq \text{End}(F(\mathcal{X}))$ be a (finite) subset. If each $\psi \in \Psi$ induces an endomorphism of $F(\mathcal{X})/R$, then $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \cup \Psi \mid \mathcal{R} \rangle$ is a (finite) L -presentation for G .*

Proof. If $\psi \in \Psi$ induces an endomorphism of $F(\mathcal{X})/R$, the normal subgroup R is ψ -invariant. Therefore, each image $r^\sigma \in F(\mathcal{X})$, with $\sigma \in (\Phi \cup \Psi)^* \setminus \Phi^*$ and $r \in \mathcal{R}$, is a relation of the group. Adding these (possibly infinitely many) relations to the group presentation does not change the isomorphism type of the group. \square

For an invariant L -presentation, we even have the following

Proposition 3.10 *Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ be a (finitely) L -presented group and let $\Psi \subseteq \text{End}(F(\mathcal{X}))$ be a (finite) subset. Then $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \cup \Psi \mid \mathcal{R} \rangle$ is a (finite) L -presentation for G if and only if each $\psi \in \Psi$ induces an endomorphism of G .*

Proof. Let $K = \langle \mathcal{Q} \cup \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle^F$ be the kernel of the free presentation $\pi: F(\mathcal{X}) \rightarrow G$. If each $\psi \in \Psi$ induces an endomorphism of $F(\mathcal{X})/K$, Proposition 3.9 shows the first assertion. If, on the other hand, the invariant L -presentations $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ and $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \cup \Psi \mid \mathcal{R} \rangle$ are L -presentations for G , each $\psi \in \Psi$ induces an endomorphism of $G = F(\mathcal{X})/K$. \square

Every substitution $\sigma \in \Phi$ of an invariant L -presentation $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ induces an endomorphism of G . However, there are possibly other endomorphisms of the free group $F(\mathcal{X})$ that will induce the same endomorphism on G . The following proposition allows us to modify a given substitution of an L -presentation:

Proposition 3.11 *Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ be a (finitely) L -presented group, $\mathcal{S} \subseteq \langle \bigcup_{\varphi \in \Phi^*} \mathcal{R}^\varphi \rangle^F$ be a (finite) subset, and let $\sigma \in \Phi$ be given. Define an endomorphism $\tilde{\sigma}$ of the free group $F = F(\mathcal{X})$ over the alphabet \mathcal{X} that is induced by the map $\tilde{\sigma}: x \mapsto x^\sigma r_x$ for each $x \in \mathcal{X}$ and some $r_x \in \mathcal{S}$. Then $\langle \mathcal{X} \mid \mathcal{Q} \mid (\Phi \setminus \{\sigma\}) \cup \{\tilde{\sigma}\} \mid \mathcal{R} \cup \mathcal{S} \rangle$ is a (finite) L -presentation for G .*

Proof. We work in the free group $F = F(\mathcal{X})$ over the alphabet \mathcal{X} and we decompose the kernel $K = \langle \mathcal{Q} \cup \bigcup_{\varphi \in \Phi^*} \mathcal{R}^\varphi \rangle^F$ of the free presentation $\pi: F \rightarrow G$ into the normal subgroups $Q = \langle \mathcal{Q} \rangle^F$ and $R = \langle \bigcup_{\varphi \in \Phi^*} \mathcal{R}^\varphi \rangle^F$ as above. Since $\mathcal{S} \subseteq \langle \bigcup_{\varphi \in \Phi^*} \mathcal{R}^\varphi \rangle^F$ holds, Proposition 3.2 yields that $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \cup \mathcal{S} \rangle$. In particular, we have that $R = \langle \bigcup_{\varphi \in \Phi^*} (\mathcal{R} \cup \mathcal{S})^\varphi \rangle^F$. Write $\Psi = (\Phi \setminus \{\sigma\}) \cup \{\tilde{\sigma}\}$. We prove this proposition by showing that the normal subgroups $R = \langle \bigcup_{\varphi \in \Phi^*} (\mathcal{R} \cup \mathcal{S})^\varphi \rangle^F$ and $\tilde{R} = \langle \bigcup_{\varphi \in \Psi^*} (\mathcal{R} \cup \mathcal{S})^\varphi \rangle^F$ coincide. For this purpose, we prove that, for each $\tilde{\delta} \in \Psi^*$ and $g \in F$, there exists $\delta \in \Phi^*$ and $h \in L = \langle \bigcup_{\varphi \in \Phi^*} \mathcal{S}^\varphi \rangle^F$ so that $g^{\tilde{\delta}} = g^\delta h$ holds. By construction, we have that $L \subseteq R$. By symmetry (as we have both $x^{\tilde{\sigma}} = x^\sigma r_x$ and $x^\sigma = x^{\tilde{\sigma}} r_x^{-1}$) the same arguments will show that, for each $\delta \in \Phi^*$ and $g \in F$, there exists $\tilde{\delta} \in \Psi^*$ and $h \in \tilde{L} = \langle \bigcup_{\varphi \in \Psi^*} \mathcal{S}^\varphi \rangle^F$ so that $g^\delta = g^{\tilde{\delta}} h$ holds. This would yield that each normal generator $s^{\tilde{\delta}} \in \tilde{R}$, with $s \in \mathcal{R} \cup \mathcal{S}$ and $\tilde{\delta} \in \Psi^*$, can be written as $s^{\tilde{\delta}} = s^\delta h$ for some $\delta \in \Phi^*$ and $h \in L \subseteq R$. In fact, $s^{\tilde{\delta}} \in \tilde{R}$ satisfies that $s^{\tilde{\delta}} = s^\delta h \in R$ and thus $\tilde{R} \subseteq R$. By symmetry, we would also obtain that $R \subseteq \tilde{R}$ holds. This clearly proves Proposition 3.11.

It therefore remains to prove that, for each $\tilde{\delta} \in \Psi^*$ and $g \in F$, there exists $\delta \in \Phi^*$ and $h \in L$ so that $g^{\tilde{\delta}} = g^\delta h$ holds. Each $g \in F$ is represented by a finite word $w_g(x_{i_1}, \dots, x_{i_n})$ over finitely many generators $\{x_{i_1}, \dots, x_{i_n}\} \subseteq \mathcal{X}$. Let $\tilde{\delta} \in \Psi^*$ and $g \in F$ be given. We prove the assertion by induction on $m = \|\tilde{\delta}\|$. If $m = 1$, then $\tilde{\delta} \in \Psi$. Moreover, we either have $\tilde{\delta} = \tilde{\sigma}$ or $\tilde{\delta} \neq \tilde{\sigma}$. If $\tilde{\delta} \neq \tilde{\sigma}$ holds, then $\tilde{\delta} \in \Phi$ and thus $g^{\tilde{\delta}} = g^\delta h$ for some $\delta \in \Phi$ and $h \in L$. Otherwise, if $\tilde{\delta} = \tilde{\sigma}$ holds, we obtain that

$$g^{\tilde{\sigma}} = w_g(x_{i_1}, \dots, x_{i_n})^{\tilde{\sigma}} = w_g(x_{i_1}^{\tilde{\sigma}}, \dots, x_{i_n}^{\tilde{\sigma}}) = w_g(x_{i_1}^\sigma r_{x_{i_1}}, \dots, x_{i_n}^\sigma r_{x_{i_n}}).$$

Conjugation in the free group F yields that the word $w_g(x_{i_1}^\sigma r_{x_{i_1}}, \dots, x_{i_n}^\sigma r_{x_{i_n}})$ can be written as $w_g(x_{i_1}^\sigma, \dots, x_{i_n}^\sigma) \cdot h$ for some $h \in \langle \mathcal{S} \rangle^F$. Thus $g^{\tilde{\sigma}} = g^\sigma \cdot h$ for some $\sigma \in \Phi$ and $h \in \langle \mathcal{S} \rangle^F \subseteq L$.

For an integer $m > 1$, assume that, for every $g \in F$ and $\tilde{\delta} \in \Psi^*$, with $\|\tilde{\delta}\| = m$, the image $g^{\tilde{\delta}} \in \tilde{R}$ satisfies that $g^{\tilde{\delta}} = g^\delta h$ for $\delta \in \Phi^*$ and some $h \in L$. Let $g \in F$ and $\tilde{\delta} \in \Psi^*$, with $\|\tilde{\delta}\| = m + 1$, be given. Then there exist $\tilde{\omega} \in \Psi$ and $\tilde{\gamma} \in \Psi^*$, with $\|\tilde{\gamma}\| = n$, so that $\tilde{\delta} = \tilde{\gamma}\tilde{\omega}$ holds. By our assumption, there exist $\gamma \in \Phi^*$ and $h \in L$ so that $g^{\tilde{\gamma}} = g^\gamma h$ holds. Thus $g^{\tilde{\delta}} = g^{\tilde{\gamma}\tilde{\omega}} = (g^\gamma h)^{\tilde{\omega}}$. If $\tilde{\omega} \neq \tilde{\sigma}$ holds, then $\tilde{\omega} \in \Phi$ and thus $\gamma\tilde{\omega} \in \Phi^*$. Moreover, by construction, the normal subgroups $L = \langle \bigcup_{\varphi \in \Phi^*} \mathcal{S}^\varphi \rangle^F$ and $\tilde{L} = \langle \bigcup_{\varphi \in \Psi^*} \mathcal{S}^\varphi \rangle^F$ are Φ^* - and Ψ^* -invariant, respectively. Thus $h^{\tilde{\omega}} \in L$ if $\tilde{\omega} \neq \tilde{\sigma}$. Therefore, the image $g^{\tilde{\delta}}$ satisfies that $g^{\tilde{\delta}} = g^{\gamma\tilde{\omega}} h^{\tilde{\omega}}$ for some $\gamma\tilde{\omega} \in \Phi^*$ and $h^{\tilde{\omega}} \in L$. It suffices to consider the case $\tilde{\omega} = \tilde{\sigma}$. The elements $g^\gamma \in F$ and $h \in F$ are represented by finite words $w_{g^\gamma}(x_{j_1}, \dots, x_{j_n})$ and $w_h(x_{k_1}, \dots, x_{k_\ell})$, respectively. Again, conjugation in the free group F yields that $w_{g^\gamma}(x_{j_1}, \dots, x_{j_n})^{\tilde{\sigma}} = w_{g^\gamma}(x_{j_1}^\sigma, \dots, x_{j_n}^\sigma) u$ and $w_h(x_{k_1}, \dots, x_{k_\ell})^{\tilde{\sigma}} = w_h(x_{k_1}^\sigma, \dots, x_{k_\ell}^\sigma) v$ with $u, v \in \langle \mathcal{S} \rangle^F$. Thus $g^{\tilde{\delta}} = g^{\tilde{\gamma}\tilde{\sigma}} = (g^\gamma h)^{\tilde{\sigma}} = (g^\gamma u)(h^\sigma v)$. In fact, we have that $g^{\tilde{\delta}} = g^{\gamma\sigma} h'$ with $\gamma\sigma \in \Phi^*$ and $h' = u h^\sigma v \in L$. Thus, for every $g \in F$ and $\tilde{\delta} \in \Psi^*$, the image $g^{\tilde{\delta}}$ satisfies that $g^{\tilde{\delta}} = g^\delta h$ with $\delta \in \Phi^*$ and $h \in L$. By symmetry, as we have both $x^{\tilde{\sigma}} = x^\sigma r_x$ and $x^\sigma = x^{\tilde{\sigma}} r_x^{-1}$, the same arguments will prove that for each $g \in F$ and

$\delta \in \Phi^*$ the image g^δ satisfies that $g^\delta = g^{\tilde{\delta}}h$ with $\tilde{\delta} \in \Psi^*$ and $h \in \tilde{L} = \langle \bigcup_{\varphi \in \Psi^*} \mathcal{S}^\varphi \rangle^F$. This finishes our proof of Proposition 3.11. \square

As a consequence of Proposition 3.11, we obtain the following

Corollary 3.12 *Let $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ be a finitely L -presented group and let $\sigma \in \Phi$ be given. Then σ induces an endomorphism of the invariantly finitely L -presented group $H = \langle \mathcal{X} \mid \emptyset \mid \Phi \mid \mathcal{R} \rangle$. If $\psi \in \text{End}(F(\mathcal{X}))$ and σ induce the same endomorphism on H , then there exists a finite subset $\mathcal{S} \subseteq F(\mathcal{X})$ so that $\langle \mathcal{X} \mid \mathcal{Q} \mid (\Phi \setminus \{\sigma\}) \cup \{\psi\} \mid \mathcal{R} \cup \mathcal{S} \rangle$ is a finite L -presentation for G .*

Proof. If σ and ψ induce the same endomorphism of H , there exists, for each $x \in \mathcal{X}$, an element $r_x \in \langle \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \rangle^F$ with $x^\psi = x^\sigma r_x$. Write $\mathcal{S} = \{r_x \mid x \in \mathcal{X}\}$. Then Proposition 3.11 yields that $G = \langle \mathcal{X} \mid \mathcal{Q} \mid (\Phi \setminus \{\sigma\}) \cup \{\psi\} \mid \mathcal{R} \cup \mathcal{S} \rangle$. \square

The transformations introduced above allow us to modify a given L -presentation of a group. In order to prove Tietze's theorem for invariantly finitely L -presented groups, we choose the following set of transformations:

Definition 3.13 *An L -Tietze transformation is a transformation that*

- (i) *adds or removes a single fixed relation (Proposition 3.1),*
- (ii) *adds or removes a single iterated relation (Proposition 3.2),*
- (iii) *adds or removes a single substitution (Proposition 3.9),*
- (iv) *adds or removes a generator together with a fixed relation (Proposition 3.5),*
- (v) *adds or removes a generator together with an iterated relation (Proposition 3.6), or that*
- (vi) *modifies a given substitution of an L -presentation (Proposition 3.11).*

4 Applications of Tietze Transformations

The transformations introduced in Section 3 allow us to prove Theorem A:

Proof of Theorem A. We use similar ideas as in the proof of Tietze's theorem in [16, Chapter II]: As each L -Tietze transformation does not change the isomorphism type of the group, two finite L -presentations define isomorphic groups if one L -presentation can be transformed into the other by a finite sequence of L -Tietze transformations. In order to prove Theorem A, it suffices to prove that two invariant finite L -presentations which define isomorphic groups can be transformed into each other by a finite sequence of L -Tietze transformations. For this purpose, we consider two invariant finite L -presentations $\langle \mathcal{X}_1 \mid \mathcal{Q}_1 \mid \Phi_1 \mid \mathcal{R}_1 \rangle$ and $\langle \mathcal{X}_2 \mid \mathcal{Q}_2 \mid \Phi_2 \mid \mathcal{R}_2 \rangle$ of a group G . By Proposition 3.4, we can assume that both $\mathcal{Q}_1 = \emptyset$ and $\mathcal{Q}_2 = \emptyset$ hold. We will construct an invariant finite L -presentation for G which can be obtained from both L -presentations by a finite sequence of L -Tietze transformations. Because each L -Tietze transformation is reversible, this shows that we can pass from one L -presentation to the other by a finite sequence of L -Tietze transformations.

Suppose that $\mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$ holds. For $i \in \{1, 2\}$, we denote by $F_i = F(\mathcal{X}_i)$ the free group over the alphabet \mathcal{X}_i and by $\pi_i: F_i \rightarrow G$ the free presentation with kernel $\ker(\pi_i) = \langle \bigcup_{\sigma \in \Phi_i^*} \mathcal{R}_i^\sigma \rangle^{F_i}$. For each $x \in \mathcal{X}_1$, we choose $w_x \in F_2$ with $x^{\pi_1} = w_x^{\pi_2}$; i.e., the element $w_x \in F_2$ is a π_2 -preimage of $x^{\pi_1} \in G$. For each $z \in \mathcal{X}_2$, we choose $w_z \in F_1$ with $z^{\pi_2} = w_z^{\pi_1}$. Define the subsets $S_1 = \{x^{-1}w_x \mid x \in \mathcal{X}_1\}$ and $S_2 = \{z^{-1}w_z \mid z \in \mathcal{X}_2\}$ of the free group $F = F(\mathcal{X}_1 \cup \mathcal{X}_2)$ over the alphabet $\mathcal{X}_1 \cup \mathcal{X}_2$. By Proposition 3.6, we can add the finitely many generators $z \in \mathcal{X}_2$ together with

the iterated relation $z^{-1}w_z \in S_2$ if we extend each substitution $\sigma \in \Phi_1$ to the free group F by

$$\tilde{\sigma}: \begin{cases} x & \mapsto x^\sigma, & \text{for each } x \in \mathcal{X}_1, \\ z & \mapsto w_z^\sigma, & \text{for each } z \in \mathcal{X}_2. \end{cases}$$

This yields the finite L -presentation

$$\langle \mathcal{X}_1 \cup \mathcal{X}_2 \mid \emptyset \mid \{\tilde{\sigma}\}_{\sigma \in \Phi_1} \mid \mathcal{R}_1 \cup \{z^{-1}w_z\}_{z \in \mathcal{X}_2} \rangle$$

for the group G . The natural homomorphisms $\pi_1: F_1 \rightarrow G$ and $\pi_2: F_2 \rightarrow G$ extend to a natural homomorphism $\pi: F \rightarrow G$ that is induced by the map

$$\pi: \begin{cases} x & \mapsto x^{\pi_1}, & \text{for each } x \in \mathcal{X}_1, \\ z & \mapsto z^{\pi_2}, & \text{for each } z \in \mathcal{X}_2. \end{cases}$$

Its kernel satisfies $\ker(\pi) = \langle \bigcup_{\sigma \in \Phi_1^*} (\mathcal{R}_1 \cup S_2)^{\tilde{\sigma}} \rangle^F$. For $x \in \mathcal{X}_1$ and $x^{-1}w_x \in S_1$, we have $x^\pi = x^{\pi_1} = w_x^{\pi_2} = w_x^\pi$ and thus $x^{-1}w_x \in \ker(\pi)$ holds. For each $r \in \mathcal{R}_2$, we have $r^\pi = r^{\pi_2} = 1$ and thus $r \in \ker(\pi)$ holds. Since the kernel $\ker(\pi)$ is $\{\tilde{\sigma} \mid \sigma \in \Phi_1\}^*$ -invariant, by construction, Proposition 3.2 yields that

$$G \cong \langle \mathcal{X}_1 \cup \mathcal{X}_2 \mid \emptyset \mid \{\tilde{\sigma}\}_{\sigma \in \Phi_1} \mid \mathcal{R}_1 \cup \mathcal{R}_2 \cup S_1 \cup S_2 \rangle.$$

As the invariant finite L -presentations $\langle \mathcal{X}_1 \mid \emptyset \mid \Phi_1 \mid \mathcal{R}_1 \rangle$ and $\langle \mathcal{X}_2 \mid \emptyset \mid \Phi_2 \mid \mathcal{R}_2 \rangle$ define isomorphic groups and every $\psi \in \Phi_2$ induces an endomorphism of the whole group, we can extend ψ to an endomorphism of the free group F over the alphabet $\mathcal{X}_1 \cup \mathcal{X}_2$ that induces the same endomorphism on G as ψ does. More precisely, for each $\psi \in \Phi_2$, we define an endomorphism of the free group F that is induced by the map

$$\tilde{\psi}: \begin{cases} z & \mapsto z^\psi, & \text{for each } z \in \mathcal{X}_2 \\ x & \mapsto w_x^\psi, & \text{for each } x \in \mathcal{X}_1 \text{ and } x^{-1}w_x \in S_1. \end{cases}$$

By construction, the normal subgroup $\langle \bigcup_{\sigma \in \Phi_1^*} (\mathcal{R}_1 \cup \mathcal{R}_2 \cup S_1 \cup S_2)^{\tilde{\sigma}} \rangle^F$ is $\tilde{\psi}$ -invariant. Thus, by Proposition 3.9, the group G satisfies that

$$G \cong \langle \mathcal{X}_1 \cup \mathcal{X}_2 \mid \emptyset \mid \{\tilde{\sigma}\}_{\sigma \in \Phi_1} \cup \{\tilde{\psi}\}_{\psi \in \Phi_2} \mid \mathcal{R}_1 \cup \mathcal{R}_2 \cup S_1 \cup S_2 \rangle. \quad (7)$$

Since the L -presentations $\langle \mathcal{X}_1 \mid \mathcal{Q}_1 \mid \Phi_1 \mid \mathcal{R}_1 \rangle$ and $\langle \mathcal{X}_2 \mid \mathcal{Q}_2 \mid \Phi_2 \mid \mathcal{R}_2 \rangle$ were finite, we have applied only finitely many L -Tietze transformations from Definition 3.13. Therefore, starting with the L -presentation $\langle \mathcal{X}_1 \mid \mathcal{Q}_1 \mid \Phi_1 \mid \mathcal{R}_1 \rangle$ we have obtained the L -presentation in Eq. (7) after finitely many steps. By symmetry, though, we would also obtain the finite L -presentation in Eq. (7) if we would have started with the finite L -presentation $\langle \mathcal{X}_2 \mid \mathcal{Q}_2 \mid \Phi_2 \mid \mathcal{R}_2 \rangle$. Since each L -Tietze transformation is reversible, we can therefore transform the finite L -presentation in Eq.(7) to the finite L -presentation $\langle \mathcal{X}_2 \mid \mathcal{Q}_2 \mid \Phi_2 \mid \mathcal{R}_2 \rangle$. This yields a finite sequence of L -Tietze transformations that allows us to transform the L -presentation $\langle \mathcal{X}_1 \mid \mathcal{Q}_1 \mid \Phi_1 \mid \mathcal{R}_1 \rangle$ to the L -presentation $\langle \mathcal{X}_2 \mid \mathcal{Q}_2 \mid \Phi_2 \mid \mathcal{R}_2 \rangle$ and vice versa. \square

Similarly, the Tietze transformations in Section 3 also allow us to prove that two arbitrary finite L -presentations could be transformed into each other by a finite sequence of Tietze transformations.

Another application of L -Tietze transformations is to prove that ‘being invariantly finitely L -presented’ is an abstract property of a group that does not depend on the generating set of the group; that is, if a group admits an invariant finite L -presentation with respect to one finite generating set, then so it does with respect to any other finite generating set. This result was already posed in [1, Proposition 2.2]. However, its proof contains a gap: Consider the invariant finite L -presentation

$$\mathfrak{G} \cong \langle \{a, b, c, d\} \mid \{a^2, b^2, c^2, d^2, bcd\} \mid \{\sigma\} \mid \{(ad)^4, (adacac)^4\} \rangle$$

from Theorem 2.2, where σ is induced by the map $a \mapsto aca$, $b \mapsto d$, $c \mapsto b$, and $d \mapsto c$. Then σ is a monomorphism of the free group $F = F(\{a, b, c, d\})$. The transformations in the proof of [1, Proposition 2.2] keep the rank of $\text{im}(\sigma)$ constant and therefore, they do not allow to prove that the Grigorchuk group admits an invariant finite L -presentation with generators $\{a, c, d\}$ as in Eq. (2). The L -Tietze transformations from Section 3 allow us to address this gap:

Proof of Theorem B. Let $\mathcal{Y} = \{y_1, \dots, y_n\}$ be an arbitrary finite generating set of the invariantly finitely L -presented group $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$. As G is invariantly L -presented, we can assume that $\mathcal{Q} = \emptyset$ holds. Since \mathcal{Y} generates G , there exists, for each $x \in \mathcal{X}$, a word $w_x(y_1, \dots, y_n)$ over the generators \mathcal{Y} so that $x =_G w_x(y_1, \dots, y_n)$ holds. Since $\mathcal{X} = \{x_1, \dots, x_m\}$ also generates G , there exists, for each $y \in \mathcal{Y}$, a word $w_y(x_1, \dots, x_m)$ so that $y =_G w_y(x_1, \dots, x_m)$ holds. Suppose that $\mathcal{X} \cap \mathcal{Y} = \emptyset$ holds. For each $\sigma \in \Phi$, define an endomorphism $\tilde{\sigma}$ of the free group E over the alphabet $\mathcal{X} \cup \mathcal{Y}$ that is induced by the map

$$\tilde{\sigma}: \begin{cases} x \mapsto x^\sigma, & \text{for each } x \in \mathcal{X}, \\ y \mapsto w_y(x_1, \dots, x_m)^\sigma, & \text{for each } y \in \mathcal{Y}. \end{cases}$$

Then, by Proposition 3.6, a finite L -presentation for the group G is given by

$$\langle \mathcal{X} \cup \mathcal{Y} \mid \emptyset \mid \{\tilde{\sigma}\}_{\sigma \in \Phi} \mid \mathcal{R} \cup \{y^{-1}w_y(x_1, \dots, x_m)\}_{y \in \mathcal{Y}} \rangle.$$

As this L -presentation is invariant, every $\tilde{\sigma}$, with $\sigma \in \Phi$, induces an endomorphism of the group G . Thus, as $x =_G w_x(y_1, \dots, y_n)$ holds, we have $x^{\tilde{\sigma}} =_G w_x(y_1, \dots, y_n)^{\tilde{\sigma}}$ for each $\sigma \in \Phi^*$. By Proposition 3.2, we have that

$$G \cong \langle \mathcal{X} \cup \mathcal{Y} \mid \emptyset \mid \{\tilde{\sigma}\}_{\sigma \in \Phi} \mid \mathcal{R} \cup \{y^{-1}w_y\}_{y \in \mathcal{Y}} \cup \{x^{-1}w_x\}_{x \in \mathcal{X}} \rangle. \quad (8)$$

Since \mathcal{Y} generates H , for each $z \in \mathcal{X} \cup \mathcal{Y}$ and $\sigma \in \Phi$, the image $z^{\tilde{\sigma}}$ is represented by a word $v_{z,\sigma}(y_1, \dots, y_n)$ over the generators \mathcal{Y} so that $z^{\tilde{\sigma}} =_G v_{z,\sigma}(y_1, \dots, y_n)$ holds. Since the L -presentation in Eq. (8) is invariant, Proposition 3.11 applies to the relation $r = (z^{\tilde{\sigma}})^{-1}v_{z,\sigma}(y_1, \dots, y_n)$ and it shows that G admits the following finite L -presentation

$$\langle \mathcal{X} \cup \mathcal{Y} \mid \emptyset \mid \{\hat{\sigma}\}_{\sigma \in \Phi} \mid \mathcal{R} \cup \{x^{-1}w_x\}_{x \in \mathcal{X}} \cup \{y^{-1}w_y\}_{y \in \mathcal{Y}} \cup \{(z^{\tilde{\sigma}})^{-1}v_{z,\sigma}\}_{z \in \mathcal{X} \cup \mathcal{Y}, \sigma \in \Phi} \rangle$$

where the substitutions $\hat{\sigma}$ are induced by the maps

$$\hat{\sigma}: z \mapsto v_{z,\sigma}(y_1, \dots, y_n), \text{ for each } z \in \mathcal{X} \cup \mathcal{Y}.$$

We use the iterated relations $x^{-1}w_x(y_1, \dots, y_n)$, with $x \in \mathcal{X}$, to replace every occurrence of $x \in \mathcal{X}$ among the iterated relations

$$\mathcal{R} \cup \{y^{-1}w_y(x_1, \dots, x_m)\}_{y \in \mathcal{Y}} \cup \{(z^{\tilde{\sigma}})^{-1}v_{z,\sigma}(y_1, \dots, y_n)\}_{z \in \mathcal{X} \cup \mathcal{Y}, \sigma \in \Phi} \quad (9)$$

by $w_x(y_1, \dots, y_n)$. This yields a finite set of relations $\tilde{\mathcal{S}}$ that can be considered as a finite subset of the free group over the alphabet \mathcal{Y} . Replacing the relations in Eq. (9) by $\tilde{\mathcal{S}}$ does not change the isomorphism type of the group. The group G satisfies that $G \cong \langle \mathcal{X} \cup \mathcal{Y} \mid \emptyset \mid \{\hat{\sigma} \mid \sigma \in \Phi\} \mid \tilde{\mathcal{S}} \cup \{x^{-1}w_x \mid x \in \mathcal{X}\} \rangle$. By Proposition 3.6, the group G is invariantly finitely L -presented by $\langle \mathcal{Y} \mid \emptyset \mid \{\hat{\sigma}\}_{\sigma \in \Phi} \mid \tilde{\mathcal{S}} \rangle$. \square

5 Finitely generated normal subgroups of finitely presented groups

In this section, we consider finitely generated *normal* subgroups of finitely presented groups. By Higman's embedding theorem [14], every finitely generated group embeds into a finitely presented group if and only if it is recursively presented. This

theorem classifies the finitely generated subgroups of a finitely presented group. The *normal* subgroups of a finitely presented group are invariantly L -presented:

Proposition 5.1 *Every normal subgroup of a finitely presented group admits an invariant L -presentation whose substitutions induce automorphisms of the subgroup. If the normal subgroup has finite index, it is invariantly finitely L -presented.*

Proof. This follows from the proof of [13, Theorem 6.1]; cf. Lemma 5.3 below. \square

The L -presentation in Lemma 5.3 below is an ascending L -presentation with finitely many substitutions and finitely many iterated relations. It has finitely many generators if and only if the subgroup has finite index. The substitutions of this L -presentation induce automorphisms of the subgroup since they are induced by conjugation in the finitely presented group.

On the other hand, as every finite L -presentation is recursive, finitely L -presented groups embed into finitely presented groups. As indicated in [4], a finitely L -presented group embeds as a normal subgroup into a finitely presented group if we assume that every substitution of the L -presentation induces an automorphism of the subgroup:

Proposition 5.2 *Every group that admits an invariant finite L -presentation, whose substitutions induce automorphisms of the group, embeds as a normal subgroup into a finitely presented group.*

Proof. If $H = \langle \mathcal{Z} \mid \emptyset \mid \{\delta_1, \dots, \delta_n\} \mid \mathcal{R} \rangle$ is invariantly finitely L -presented so that each δ_i induces an automorphism of H , the base group H embeds into the HNN-extension G_1 relative to the isomorphism $\delta_1: H \rightarrow H$ which is induced by the substitution δ_1 . The HNN-extension G_1 is given by the presentation $G_1 = \langle \mathcal{Z} \cup \{t_1\} \mid \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \cup \{t_1^{-1} z t_1 = z^{\delta_1} \mid z \in \mathcal{Z}\} \rangle$ where $\Phi = \{\delta_1, \dots, \delta_n\}$. Denote by H_1 the image of H in G_1 . Then δ_2 induces an automorphism of the subgroup $H_1 \leq G_1$. Thus we can form the HNN-extension G_2 relative to the isomorphism $\delta_2: H_1 \rightarrow H_1$. As the base group G_1 embeds into the HNN-extension G_2 , the subgroup H_1 embeds into G_2 as well. Iterating this process, we obtain a group $G_n = \langle \mathcal{Z} \cup \{t_1, \dots, t_n\} \mid \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \cup \{t_i^{-1} z t_i = z^{\delta_i} \mid 1 \leq i \leq n, z \in \mathcal{Z}\} \rangle$ in which H embeds. Tietze transformations that replace every δ_i -image z^{δ_i} by the conjugate $t_i^{-1} z t_i$ in the relations $\bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma$ eventually show that $G_n = \langle \mathcal{Z} \cup \{t_1, \dots, t_n\} \mid \mathcal{R} \cup \{t_i^{-1} z t_i = z^{\delta_i} \mid 1 \leq i \leq n, z \in \mathcal{Z}\} \rangle$ is finitely presented. The invariantly finitely L -presented group H embeds into this finitely presented group by identifying the generator in \mathcal{Z} . The image of H in G_n is obviously a normal subgroup of G_n . \square

In the following, we use the constructions from [4] to prove Theorem C. Since every normal subgroup of a finitely presented group admits an invariant L -presentation with finitely many substitutions and finitely many iterated relations, it suffices to show that the L -presentation in Lemma 5.3 below could be transformed into an invariant finite L -presentation. For this purpose, though, we need to eliminate (possibly) infinitely many generators from the L -presentation and we need to modify finitely many substitutions. However, Proposition 3.11 adds iterated relations for each modification of a substitution. Hence, we need to ensure that this process still gives a *finite* L -presentation. In the following, we generalize the constructions from [4]:

5.1 Preliminaries

Let G be a finitely presented group and let $H \trianglelefteq G$ be a finitely generated normal subgroup. Then G/H is finitely presented. Moreover, if $H = \langle a_1, \dots, a_m \rangle$ and $G/H = \langle s_1 H, \dots, s_n H \rangle$ hold, there exists a finite presentation $\langle \{a_1, \dots, a_m, s_1, \dots, s_n\} \mid \mathcal{R} \rangle$ for G . The proof of [13, Theorem 6.1] yields the following

Lemma 5.3 *Let $\langle \{a_1, \dots, a_m, s_1, \dots, s_n\} \mid \mathcal{R} \rangle$ be a finite presentation for G and write $\mathcal{S} = \{s_1^{\pm 1}, \dots, s_n^{\pm 1}\}$. If \mathcal{T} is a Schreier transversal for $H = \langle a_1, \dots, a_m \rangle$ in G and \mathcal{Y} are the Schreier generators of H , then H is invariantly L -presented by*

$$\langle \mathcal{Y} \mid \emptyset \mid \{\delta_x \mid x \in \mathcal{S}\} \mid \mathcal{R}^\tau \rangle$$

where δ_x denotes the endomorphism of the free group $F(\mathcal{Y})$ that is induced by conjugation with $x \in \mathcal{S}$ and τ denotes the Reidemeister-rewriting.

Proof. This follows from the Reidemeister-Schreier theorem, see [16, Section II.4] and the proof of [13, Theorem 6.1]. Clearly, one can always omit the endomorphisms δ_x with $x \in \{a_1, \dots, a_m\}$ as they give inner automorphisms of the subgroup H . \square

Since \mathcal{S} and \mathcal{R} are finite, the L -presentation in Lemma 5.3 is finite if and only if H has finite index in G ; in this case \mathcal{Y} is finite. Finite index subgroups of finitely L -presented groups have been studied in [13]. It was shown that each normal subgroup of a finitely presented group with finite index is invariantly finitely L -presented. In the following, we therefore assume that $[G : H] = \infty$ holds.

The strategy in the proof of Theorem C will be as follows: Our choice of the generating set of the finitely presented group allows us to assume that H 's generators $\mathcal{Z} = \{a_1, \dots, a_m\}$ are Schreier generators of H . We therefore obtain an embedding $\chi: F(\mathcal{Z}) \rightarrow F(\mathcal{Y})$ and we will construct an epimorphism $\gamma: F(\mathcal{Y}) \rightarrow F(\mathcal{Z})$ so that the free presentation $\pi: F(\mathcal{Y}) \rightarrow H$ that is given by the L -presentation in Lemma 5.3 satisfies $\gamma\chi\pi = \pi$. Since the L -presentation in Lemma 5.3 is invariant, there exists, for each $\sigma \in \Phi = \{\delta_x \mid x \in \mathcal{S}\}$, an endomorphism $\hat{\sigma} \in \text{End}(H)$ so that $\sigma\pi = \pi\hat{\sigma}$ holds. In general, we cannot assume that there also exists an endomorphism $\tilde{\sigma} \in \text{End}(F(\mathcal{Z}))$ so that $\sigma\gamma = \gamma\tilde{\sigma}$ holds. Therefore, we will construct a normal subgroup $N \trianglelefteq F(\mathcal{Z})$ so that $\psi: F(\mathcal{Z}) \rightarrow F(\mathcal{Z})/N$, $g \mapsto gN$ yields the existence of $\bar{\sigma} \in \text{End}(F(\mathcal{Z})/N)$ with $\sigma\gamma\psi = \gamma\psi\bar{\sigma}$. These constructions will give the following commutative diagram:

$$\begin{array}{ccc}
 & F(\mathcal{Z}) & \xrightarrow{\psi} F(\mathcal{Z})/N \\
 \gamma \nearrow & \downarrow \chi\pi & \nwarrow \\
 F(\mathcal{Y}) & \xrightarrow{\pi} & H
 \end{array}$$

δ_x (loop on $F(\mathcal{Y})$) $\hat{\delta}_x$ (loop on H) $\bar{\delta}_x$ (loop on $F(\mathcal{Z})/N$)

In the special cases of Theorem C and Theorem D, we are able to prove that $F(\mathcal{Z})/N$ is invariantly finitely L -presented and so is the subgroup H . The normal subgroup N will be generated, as a normal subgroup, by the iterated relations that Proposition 3.11 adds when modifying the substitutions of the L -presentation in Lemma 5.3. These relations were omitted in [4]. It is not clear whether or not these relations are necessary to define the subgroup H .

In the remainder of this section, we generalize the constructions from [4] to obtain the commutative diagram above. The generating set $\mathcal{X} = \{a_1, \dots, a_m, s_1, \dots, s_n\}$ of the finitely presented group G yields that the generators $\mathcal{Z} = \{a_1, \dots, a_m\}$ are Schreier generators of H . Hence, there exists a natural embedding $\chi: F(\mathcal{Z}) \rightarrow F(\mathcal{Y})$ which is induced by embedding the generators \mathcal{Z} into \mathcal{Y} . It suffices to remove the Schreier generators $\mathcal{Y} \setminus \mathcal{Z}$ from the invariant L -presentation in Lemma 5.3. Since H is generated by $\mathcal{Z} = \{a_1, \dots, a_m\}$, every $y \in \mathcal{Y}$ can be represented, as an element of H , by a word over \mathcal{Z} . This yields an epimorphism $\gamma: F(\mathcal{Y}) \rightarrow F(\mathcal{Z})$ which maps

every $y \in \mathcal{Y}$ to a word $y^\gamma \in F(\mathcal{Z})$ over the alphabet \mathcal{Z} that represents the same element in H ; i.e., we have

$$\{y^{-1}y^{\gamma\chi} \mid y \in \mathcal{Y} \setminus \mathcal{Z}\} \subseteq \ker(\pi), \quad (10)$$

where $\pi: F(\mathcal{Y}) \rightarrow H$ denotes the free presentation from Lemma 5.3. Note that Eq. (10) yields that $\iota = \chi\pi$ defines an epimorphism $\iota: F(\mathcal{Z}) \rightarrow H$ with $\gamma\iota = \pi$. The following lemma generalizes [4, Lemma 4].

Lemma 5.4 *If $H \cong \langle \mathcal{Y} \mid \mathcal{S} \rangle$ and $\gamma: F(\mathcal{Y}) \rightarrow F(\mathcal{Z})$ is an epimorphism so that*

$$\begin{array}{ccc} & & F(\mathcal{Z}) \\ & \nearrow \gamma & \downarrow \iota \\ F(\mathcal{Y}) & \xrightarrow{\pi} & H \end{array}$$

commutes, $\langle \mathcal{Z} \mid \mathcal{S}^\gamma \rangle$ is a presentation for H .

Proof. Since $\pi = \gamma\iota$ is onto, it suffices to prove that $\ker(\iota) = \langle \mathcal{S}^\gamma \rangle^{F(\mathcal{Z})}$ holds. For $r \in \mathcal{S}$, we have that $r^{\gamma\iota} = r^\pi = 1$ and so $r^\gamma \in \ker(\iota)$. Thus $\langle \mathcal{S}^\gamma \rangle^{F(\mathcal{Z})} \subseteq \ker(\iota)$. If $g \in \ker(\iota)$ holds, there exists $h \in F(\mathcal{Y})$ with $h^\gamma = g$ as γ is surjective. Then $h^\pi = h^{\gamma\iota} = g^\iota = 1$ and $h \in \ker(\pi) = \langle \mathcal{S} \rangle^{F(\mathcal{Y})}$. Thus $g = h^\gamma \in \langle \mathcal{S}^\gamma \rangle^{F(\mathcal{Z})}$. \square

Thus, by Lemma 5.3 and Lemma 5.4, the subgroup H has a presentation of the form

$$H = \langle \mathcal{Z} \mid \{(r^{\tau\sigma})^\gamma \mid r \in \mathcal{R}, \sigma \in \Phi^*\} \rangle$$

where $\Phi = \{\delta_x \mid x \in \mathcal{S}\}$ and τ denotes the Reidemeister rewriting. This presentation can be considered as a finite L -presentation if, for each $\sigma \in \Phi$, there exists an endomorphism $\tilde{\sigma} \in \text{End}(F(\mathcal{Z}))$ with $\sigma\gamma = \gamma\tilde{\sigma}$. The following lemma yields the existence of such endomorphisms $\tilde{\sigma} \in \text{End}(F(\mathcal{Z}))$:

Lemma 5.5 *For groups L and M , an epimorphism $\pi: L \rightarrow M$, and an endomorphism $\delta \in \text{End}(L)$, there exists a (unique) endomorphism $\Delta \in \text{End}(M)$ with $\delta\pi = \pi\Delta$ if and only if $\ker(\pi)^\delta \subseteq \ker(\pi)$ holds.*

Proof. The proof is straightforward. \square

Therefore, if the kernel $\ker(\gamma)$ is σ -invariant, for each $\sigma \in \Phi$, the subgroup H would be invariantly finitely L -presented by $\langle \mathcal{Z} \mid \emptyset \mid \{\tilde{\delta}_x \mid \delta_x \in \Phi\} \mid \mathcal{R}^{\tau\gamma} \rangle$. In general, though, we cannot assume that each $\sigma \in \Phi$ leaves the kernel $\ker(\gamma)$ invariant. If we consider the natural embedding $\chi: F(\mathcal{Z}) \rightarrow F(\mathcal{Y})$ that is induced by embedding the generators \mathcal{Z} into \mathcal{Y} , the kernel $\ker(\gamma)$ satisfies

Lemma 5.6 *If $\chi: F(\mathcal{Z}) \rightarrow F(\mathcal{Y})$ is an embedding with $\gamma\chi|_{\mathcal{Z}} = \text{id}_{\mathcal{Z}}$, then $\chi\gamma = \text{id}_{F(\mathcal{Z})}$ and $\ker(\gamma) = \langle \{y^{-1}y^{\gamma\chi} \mid y \in \mathcal{Y} \setminus \mathcal{Z}\} \rangle^{F(\mathcal{Y})}$ hold.*

Proof. Since $\gamma\chi|_{\mathcal{Z}} = \text{id}_{\mathcal{Z}}$ holds, the map $\gamma\chi$ induces the identity on the free subgroup $E = \langle \mathcal{Z} \rangle \leq F(\mathcal{Y})$. For $g \in F(\mathcal{Z})$, we have $g^\chi \in E$ and $g^{\chi\gamma} = g^\chi$. Thus $(g^{-1}g^{\chi\gamma})^\chi = 1$ and, as χ is injective, we have $g^{-1}g^{\chi\gamma} = 1$ or

$$\chi\gamma = \text{id}_{F(\mathcal{Z})}. \quad (11)$$

For each $y \in \mathcal{Y} \setminus \mathcal{Z}$, we have that $(y^{-1}y^{\gamma\chi})^\gamma = y^{-\gamma}y^{\gamma\chi\gamma} = y^{-\gamma}y^\gamma = 1$. Therefore $N = \langle \{y^{-1}y^{\gamma\chi} \mid y \in \mathcal{Y} \setminus \mathcal{Z}\} \rangle^{F(\mathcal{Y})}$ satisfies that $N \subseteq \ker(\gamma)$. Let $g \in \ker(\gamma)$ be given. Then $g \in F(\mathcal{Y})$ is represented by a finite word $w(y_{i_1}, \dots, y_{i_n}, a_1, \dots, a_m)$ with $\{y_{i_1}, \dots, y_{i_n}\} \subseteq \mathcal{Y} \setminus \mathcal{Z}$. Modulo the normal subgroup N , we can replace every occurrence of $y \in \mathcal{Y} \setminus \mathcal{Z}$ by $y^{\gamma\chi} \in E$; i.e., we have $g = w(y_{i_1}, \dots, y_{i_n}, a_1, \dots, a_m) =$

$w(y_{i_1}^{\gamma\chi}, \dots, y_{i_n}^{\gamma\chi}, a_1, \dots, a_m) \cdot h$ for some $h \in N$. As $g \in \ker(\gamma)$ and $h \in N \subseteq \ker(\gamma)$ hold, we have

$$1 = g^{\gamma\chi} = w(y_{i_1}^{\gamma\chi\gamma\chi}, \dots, y_{i_n}^{\gamma\chi\gamma\chi}, a_1^{\gamma\chi}, \dots, a_m^{\gamma\chi}) \cdot h^{\gamma\chi} = w(y_{i_1}^{\gamma\chi}, \dots, y_{i_n}^{\gamma\chi}, a_1, \dots, a_m) \cdot 1.$$

Similarly, modulo the normal subgroup N , we can replace every occurrence of $y^{\gamma\chi}$ by y . There exists $k \in N$ with $1 = w(y_{i_1}^{\gamma\chi}, \dots, y_{i_n}^{\gamma\chi}, a_1, \dots, a_m) = w(y_{i_1}, \dots, y_{i_n}, a_1, \dots, a_m) \cdot k = g \cdot k$. Thus $g \in N$ and $N = \ker(\gamma)$. \square

Even though $\delta_x \in \Phi$ may not translate directly to $\tilde{\delta}_x \in \text{End}(F(\mathcal{Z}))$, there exists a normal subgroup $N \trianglelefteq F(\mathcal{Z})$ and a homomorphism $\psi: F(\mathcal{Z}) \rightarrow F(\mathcal{Z})/N$, $g \mapsto gN$ so that $\ker(\gamma\psi)^{\delta_x} \subseteq \ker(\gamma\psi)$ holds: For each $\delta_x \in \Phi$, define $\tilde{\delta}_x = \chi\delta_x\gamma \in \text{End}(F(\mathcal{Z}))$. Consider the normal subgroup

$$N = \left\langle \bigcup_{\sigma \in \tilde{\Phi}^*} \left(\{ (y^{-1}y^{\gamma\chi})^{\delta_x\gamma} \}_{y \in \mathcal{Y} \setminus \mathcal{Z}, x \in \mathcal{S}} \right)^\sigma \right\rangle^{F(\mathcal{Z})} \quad (12)$$

where $\tilde{\Phi} = \{\tilde{\delta}_x \mid \delta_x \in \Phi\}$. By construction, $N^{\tilde{\delta}_x} \subseteq N$ and thus there exists a unique endomorphism $\bar{\delta}_x: F(\mathcal{Z})/N \rightarrow F(\mathcal{Z})/N$, $gN \mapsto g^{\tilde{\delta}_x}N$ with $\tilde{\delta}_x\psi = \psi\bar{\delta}_x$. The normal subgroup N allows us to translate $\delta_x \in \Phi$ to $\bar{\delta}_x \in \text{End}(F(\mathcal{Z})/N)$ with $\delta_x\gamma\psi = \gamma\psi\bar{\delta}_x$:

Lemma 5.7 *For each $x \in \mathcal{S}$, we have that $\ker(\gamma\psi)^{\delta_x} \subseteq \ker(\gamma\psi)$.*

Proof. The kernel $\ker(\gamma\psi) = \ker(\gamma)N^{\gamma^{-1}}$ satisfies that

$$\ker(\gamma\psi) = \left\langle \left\{ y^{-1}y^{\gamma\chi} \right\}_{y \in \mathcal{Y} \setminus \mathcal{Z}} \cup \bigcup_{\tilde{\sigma} \in \tilde{\Phi}^*} \left\{ (y^{-1}y^{\gamma\chi})^{\delta_x\gamma\tilde{\sigma}\chi} \right\}_{y \in \mathcal{Y} \setminus \mathcal{Z}, x \in \mathcal{S}} \right\rangle^{F(\mathcal{Y})}.$$

The generator $(y^{-1}y^{\gamma\chi})^{\delta_x\gamma\tilde{\sigma}\chi}$ is mapped by $\delta_x\gamma$ to $(y^{-1}y^{\gamma\chi})^{\delta_x\gamma\tilde{\sigma}\chi\delta_x\gamma} = (y^{-1}y^{\gamma\chi})^{\delta_x\gamma\tilde{\sigma}\tilde{\delta}_x} \in N$ while $y^{-1}y^{\gamma\chi}$ is mapped to $(y^{-1}y^{\gamma\chi})^{\delta_x\gamma} \in N$. \square

The endomorphisms $\delta_x \in \text{End}(F(\mathcal{Y}))$, $\tilde{\delta}_x \in \text{End}(F(\mathcal{Z}))$, and $\bar{\delta}_x \in \text{End}(F(\mathcal{Z})/N)$ also satisfy that

$$\tilde{\delta}_x\psi = \chi\delta_x\gamma\psi = \chi\gamma\psi\bar{\delta}_x = \psi\bar{\delta}_x. \quad (13)$$

Since the L -presentation in Lemma 5.3 is invariant, there exists $\hat{\delta}_x \in \text{End}(H)$ with $\delta_x\pi = \pi\hat{\delta}_x$. The subgroup H is a homomorphic image of $F(\mathcal{Z})/N$:

Lemma 5.8 *Let $\iota: F(\mathcal{Z}) \rightarrow H$, $g \mapsto g^{\chi\pi}$ be given. Then $\gamma\iota = \pi$ and $N \leq \ker(\iota)$.*

Proof. The first assertion follows from the definition of γ in Eq. (10) above. For $\delta_x \in \Phi$, we have $\tilde{\delta}_x\iota = \chi\delta_x\gamma\iota = \chi\delta_x\pi = \chi\pi\hat{\delta}_x = \iota\hat{\delta}_x$ and $\gamma\chi\pi = \gamma\iota = \pi$. Thus $(y^{-1}y^{\gamma\chi})^\pi = y^{-\pi}y^{\gamma\chi\pi} = y^{-\pi}y^\pi = 1$. For $\tilde{\sigma} \in \tilde{\Phi}^*$ with $\tilde{\sigma} = \tilde{\delta}_{x_1} \cdots \tilde{\delta}_{x_n}$ we therefore obtain

$$\delta_x\gamma\tilde{\sigma}\iota = \delta_x\gamma\tilde{\delta}_{x_1} \cdots \tilde{\delta}_{x_n}\iota = \delta_x\gamma\iota\hat{\delta}_{x_1} \cdots \hat{\delta}_{x_n} = \delta_x\pi\hat{\delta}_{x_1} \cdots \hat{\delta}_{x_n} = \pi\hat{\delta}_x\hat{\delta}_{x_1} \cdots \hat{\delta}_{x_n}.$$

Hence, for each $\tilde{\sigma} \in \tilde{\Phi}^*$, $y \in \mathcal{Y} \setminus \mathcal{Z}$, and $x \in \mathcal{X}$, the generator $(y^{-1}y^{\gamma\chi})^{\delta_x\gamma\tilde{\sigma}} \in N$ satisfies $(y^{-1}y^{\gamma\chi})^{\delta_x\gamma\tilde{\sigma}\iota} = (y^{-1}y^{\gamma\chi})^{\pi\hat{\delta}_x\hat{\delta}_{x_1} \cdots \hat{\delta}_{x_n}} = 1$ as $y^{-1}y^{\gamma\chi} \in \ker(\pi)$ holds. Therefore $N \subseteq \ker(\iota)$ holds. \square

By Lemma 5.8, the homomorphism $\varphi: F(\mathcal{Z})/N \rightarrow H$, $gN \mapsto g'$ is well-defined and it satisfies that $\psi\varphi = \iota$. We have obtained the following diagram:

$$\begin{array}{ccc}
 & \begin{array}{c} \bar{\delta}_x \\ \curvearrowright \\ F(\mathcal{Z}) \end{array} & \begin{array}{c} \bar{\delta}_x \\ \curvearrowright \\ F(\mathcal{Z})/N \end{array} \\
 & \downarrow \psi & \\
 & F(\mathcal{Z}) & \\
 \begin{array}{c} \gamma \\ \nearrow \\ F(\mathcal{Y}) \end{array} & \begin{array}{c} \chi \\ \searrow \\ H \end{array} & \begin{array}{c} \iota = \chi\pi \\ \nearrow \\ H \end{array} \\
 \begin{array}{c} \delta_x \\ \curvearrowright \\ F(\mathcal{Y}) \end{array} & \xrightarrow{\pi} & \begin{array}{c} \hat{\delta}_x \\ \curvearrowright \\ H \end{array}
 \end{array}$$

By construction, $F(\mathcal{Z})/N$ is invariantly L -presented by

$$F(\mathcal{Z})/N \cong \langle \mathcal{Z} \mid \emptyset \mid \{\tilde{\delta}_x\}_{\delta_x \in \Phi} \mid \{(y^{-1}y^{\gamma\chi})^{\delta_x\gamma}\}_{y \in \mathcal{Y} \setminus \mathcal{Z}, \delta_x \in \Phi} \rangle.$$

If $[G : H] = \infty$ holds, $|\mathcal{Y} \setminus \mathcal{Z}|$ is infinite. Therefore, the latter L -presentation is finite if and only if $[G : H]$ is finite. Our strategy in the proof of Theorem C uses the following

Lemma 5.9 *If there exists a finite set $\mathcal{U} \subseteq F(\mathcal{Z})$ with $F(\mathcal{Z})/N \cong \langle \mathcal{Z} \mid \emptyset \mid \tilde{\Phi} \mid \mathcal{U} \rangle$, then H is invariantly finitely L -presented.*

Proof. The kernel of $\varphi: F(\mathcal{Z})/N \rightarrow H$ is generated by the images $r^{\tau\sigma\gamma\psi} = r^{\tau\gamma\psi\sigma}$ with $\sigma \in \Phi^*$ and $r \in \mathcal{R}$. If $\langle \mathcal{Z} \mid \emptyset \mid \tilde{\Phi} \mid \mathcal{U} \rangle$ is an invariant finite L -presentation for $F(\mathcal{Z})/N$, then H is invariantly finitely L -presented by $\langle \mathcal{Z} \mid \emptyset \mid \tilde{\Phi} \mid \mathcal{U} \cup \mathcal{R}^{\tau\gamma} \rangle$. \square

5.2 Proofs of Theorem C and Theorem D

In this section, we prove Theorem C and Theorem D:

Proof of Theorem C. Our strategy in the proof of Theorem C is to construct a normal subgroup $N \trianglelefteq F(\mathcal{Z})$ and to prove that $F(\mathcal{Z})/N$ is invariantly finitely L -presented. Then Lemma 5.9 applies and it shows that $H \leq G$ is invariantly finitely L -presented.

Since G is finitely presented, G/H is finitely generated. Moreover, as G splits over H , there exists $s_1, \dots, s_n \in G$ so that $G/H = \langle s_1H, \dots, s_nH \rangle$ and $S = \langle s_1, \dots, s_n \rangle$ satisfies that $S \cap H = \{1\}$; i.e., $G \cong H \rtimes S$ holds. Because H is finitely generated, there exist $a_1, \dots, a_m \in H$ so that $H = \langle a_1, \dots, a_m \rangle$ holds. Then $G = \langle a_1, \dots, a_m, s_1, \dots, s_n \rangle$ holds and there exists a finite set of relations \mathcal{R} with $G \cong \langle \{a_1, \dots, a_m, s_1, \dots, s_n\} \mid \mathcal{R} \rangle$. Write $\mathcal{S} = \{s_1^{\pm 1}, \dots, s_n^{\pm 1}\}$ and $\mathcal{X} = \{a_1, \dots, a_m, s_1, \dots, s_n\}$. Clearly, we can choose a Schreier transversal $\mathcal{T} \subseteq \mathcal{S}^*$ whose elements are words over the alphabet \mathcal{S} . This yields the Schreier generators

$$\begin{aligned}
 a_{\ell,t} = \gamma(t, a_\ell) &= ta_\ell(\overline{ta_\ell})^{-1} = ta_\ell t^{-1}, \\
 s_{\ell,t} = \gamma(t, s_\ell) &= ts_\ell(\overline{ts_\ell})^{-1},
 \end{aligned}$$

with $t \in \mathcal{T}$. Then $\{s_{\ell,t} \mid 1 \leq \ell \leq n, t \in \mathcal{T}\} \subseteq \mathcal{S}^*$. By Lemma 5.3, the subgroup H is invariantly L -presented by $\langle \mathcal{Y} \mid \emptyset \mid \{\delta_s \mid s \in \mathcal{S}\} \mid \mathcal{R}^\tau \rangle$ where

$$\mathcal{Y} = \{a_{\ell,t} \mid t \in \mathcal{T}, 1 \leq \ell \leq m\} \cup \{s_{\ell,t} \neq 1 \mid t \in \mathcal{T}, 1 \leq \ell \leq n\}$$

and δ_s denotes the endomorphism of $F(\mathcal{Y})$ that is induced by conjugation with $s \in \mathcal{S}$. Write $S = \langle s_1, \dots, s_n \rangle \leq F(\mathcal{X})$ and $E = \langle a_1, \dots, a_m \rangle \leq F(\mathcal{X})$. Let $K \trianglelefteq F(\mathcal{X})$ be the kernel of G 's free presentation $F(\mathcal{X}) \rightarrow G$. Then $EK = \langle \mathcal{Y} \rangle$ and

$S \cap EK = \langle s_{\ell,t} \neq 1 \mid 1 \leq \ell \leq n, t \in \mathcal{T} \rangle$ are freely generated. For each $s \in \mathcal{S}$, the subgroup $S \cap EK$ is δ_s -invariant since $S \cap EK \triangleleft S$ holds. Because G splits over H , we have $S \cap H = \{1\}$. Thus the generators $s_{\ell,t} \in S \cap EK$ are contained in the kernel of the free presentation $\pi: F(\mathcal{Y}) \rightarrow H$ which is given by H 's invariant L -presentation above. Define $\mathcal{Z} = \{a_1, \dots, a_m\}$ and an embedding

$$\chi: F(\mathcal{Z}) \rightarrow F(\mathcal{Y}), a_\ell \mapsto a_{\ell,1}$$

where $1 \in \mathcal{T}$ denotes the trivial element in the Schreier transversal \mathcal{T} . For $s \in \mathcal{S}$ and $a_\ell \in \mathcal{Z}$, we choose a representative $a_\ell^{\chi \delta_s \gamma} \in F(\mathcal{Z})$ with

$$a_\ell^{-\chi \delta_s} (a_\ell^{\chi \delta_s \gamma})^\chi \in \ker(\pi). \quad (14)$$

For $s \in \mathcal{S}$, let $\tilde{\delta}_s \in F(\mathcal{Z})$ be induced by the map $a_\ell \mapsto a_\ell^{\chi \delta_s \gamma}$ and define $\iota: F(\mathcal{Z}) \rightarrow H$ by $\iota = \chi\pi$. Then Eq. (14) yields that $\tilde{\delta}_s \iota = \iota \hat{\delta}_s$. In the following, we write $\tilde{\delta}_t = \tilde{\delta}_{x_1} \cdots \tilde{\delta}_{x_n}$ if $t = x_1 \cdots x_n \in \mathcal{S}^*$ and each $x_i \in \mathcal{S}$. Moreover, we write X for x^{-1} and T for t^{-1} . This yields that $a_{\ell,1}^{\delta_T} = t a_\ell T = a_{\ell,t}$. Let $\gamma: F(\mathcal{Y}) \rightarrow F(\mathcal{Z})$ be induced by the map

$$\gamma: \begin{cases} a_{\ell,t} \mapsto a_\ell^{\tilde{\delta}_T}, & \text{for each } 1 \leq \ell \leq m \text{ and } t \in \mathcal{T}, \\ s_{\ell,t} \mapsto 1, & \text{for each } 1 \leq \ell \leq n \text{ and } t \in \mathcal{T}. \end{cases}$$

For each $1 \leq \ell \leq m$, $1 \leq k \leq n$, and $t \in \mathcal{T}$, this yields

$$(a_{\ell,t})^{\gamma \iota} = a_\ell^{\tilde{\delta}_T \iota} = a_\ell^{\iota \hat{\delta}_T} = a_\ell^{\chi \pi \hat{\delta}_T} = a_{\ell,1}^{\delta_T \pi} = a_{\ell,t}^\pi \quad \text{and} \quad (s_{k,t})^{\gamma \iota} = 1^\iota = 1 = (s_{k,t})^\pi.$$

Thus $\gamma \iota = \pi$. Define the normal subgroup

$$N = \left\langle \bigcup_{\tilde{\sigma} \in \tilde{\Phi}^*} \left(\{ (y^{-1} y^{\gamma \chi})^{\delta_s \gamma} \}_{y \in \mathcal{Y} \setminus \mathcal{Z}, s \in \mathcal{S}} \right)^{\tilde{\sigma}} \right\rangle^{F(\mathcal{Y})}$$

where $\tilde{\Phi} = \{\tilde{\delta}_s \mid s \in \mathcal{S}\}$. For $t \in \mathcal{T}$ and $s \in \mathcal{S}$, we have that

$$(s_{\ell,t}^{-1} s_{\ell,t}^{\gamma \chi})^{\delta_s \gamma} = s_{\ell,t}^{-\delta_s \gamma} (s_{\ell,t}^{\gamma})^{\tilde{\delta}_s} = 1$$

as the subgroup $S \cap EK = \langle s_{\ell,t} \mid t \in \mathcal{T}, 1 \leq \ell \leq n \rangle$ is δ_s -invariant and it is contained in the kernel of γ . This yields that

$$N = \left\langle \bigcup_{\tilde{\sigma} \in \tilde{\Phi}^*} \left(\{ (a_{\ell,t}^{-1} a_{\ell,t}^{\gamma \chi})^{\delta_s \gamma} \}_{1 \leq \ell \leq m, t \in \mathcal{T} \setminus \{1\}, s \in \mathcal{S}} \right)^{\tilde{\sigma}} \right\rangle^{F(\mathcal{Z})}.$$

For $t \in \mathcal{T}$ and $x \in \mathcal{S}$ with $xt \in \mathcal{T}$, we also have that

$$(a_{\ell,t}^{-1} a_{\ell,t}^{\gamma \chi})^{\delta_x \gamma} = a_{\ell,t}^{-\delta_x \gamma} a_{\ell,t}^{\gamma \tilde{\delta}_x} = a_{\ell,xt}^{-\gamma} a_{\ell,t}^{\gamma \tilde{\delta}_x} = a_\ell^{-\tilde{\delta}_T x} a_\ell^{\tilde{\delta}_T \tilde{\delta}_x} = 1.$$

It therefore suffices to consider the generators $(a_{\ell,t}^{-1} a_{\ell,t}^{\gamma \chi})^{\delta_x \gamma} \in N$ with $1 \leq \ell \leq m$, $t \in \mathcal{T}$, and $x \in \mathcal{S}$ but $xt \notin \mathcal{T}$. Since $G/H \cong S/S \cap EK$ is a finitely presented group, there exists a finite monoid presentation

$$S/S \cap EK \cong \langle \mathcal{S} \mid (U_1, V_1), \dots, (U_p, V_p) \rangle.$$

The monoid congruence \sim induced by this presentation is the reflexive, symmetric, and transitive closure of the binary relation \sim that is defined by $x \sim y$ if there exist $A, B \in \mathcal{S}^*$ and $1 \leq i \leq p$ so that $x = AU_i B$ and $y = AV_i B$ hold. Define

$$M = \left\langle \bigcup_{\tilde{\sigma} \in \tilde{\Phi}^*} \left(\{ (a_\ell^{-\tilde{\delta}_{U_i}} a_\ell^{\tilde{\delta}_{V_i}}) \}_{1 \leq \ell \leq m, 1 \leq i \leq p} \right)^{\tilde{\sigma}} \right\rangle^{F(\mathcal{Z})}.$$

Suppose that $u \sim v$ holds. Then there exist $A_i, B_i, L_i \in \mathcal{S}^*$ so that $u = L_1 \sim \dots \sim L_q = v$ with $L_i = A_i U_{\ell_i} B_i$ and $L_{i+1} = A_i V_{\ell_i} B_i$ (or $L_i = A_i V_{\ell_i} B_i$ and $L_{i+1} = A_i U_{\ell_i} B_i$). Note that

$$a_\ell^{\tilde{\delta}_{A_i} \tilde{\delta}_{U_{\ell_i}} \tilde{\delta}_{B_i}} = (a_\ell^{\tilde{\delta}_{A_i}})^{\tilde{\delta}_{U_{\ell_i}} \tilde{\delta}_{B_i}} = w_\ell(a_1, \dots, a_m)^{\tilde{\delta}_{U_{\ell_i}} \tilde{\delta}_{B_i}} = w_\ell(a_1^{\tilde{\delta}_{U_{\ell_i}}}, \dots, a_m^{\tilde{\delta}_{U_{\ell_i}}})^{\tilde{\delta}_{B_i}}$$

for some word $w_\ell(a_1, \dots, a_m) = a_\ell^{\tilde{\delta}_{A_i}} \in F(\mathcal{Z})$. The normal subgroup M yields that

$$(a_\ell^{\tilde{\delta}_{A_i}})^{\tilde{\delta}_{U_{\ell_i}}} = w_\ell(a_1^{\tilde{\delta}_{U_{\ell_i}}}, \dots, a_m^{\tilde{\delta}_{U_{\ell_i}}}) = w_\ell(a_1^{\tilde{\delta}_{V_{\ell_i}}}, \dots, a_m^{\tilde{\delta}_{V_{\ell_i}}}) \cdot h = a_\ell^{\tilde{\delta}_{A_i} \tilde{\delta}_{V_{\ell_i}}} \cdot h$$

for some $h \in M$. By construction, M is $\tilde{\Phi}^*$ -invariant and thus

$$a_\ell^{-\tilde{\delta}_{A_i} \tilde{\delta}_{V_{\ell_i}} \tilde{\delta}_{B_i}} a_\ell^{\tilde{\delta}_{A_i} \tilde{\delta}_{U_{\ell_i}} \tilde{\delta}_{B_i}} = h^{\tilde{\delta}_{B_i}} \in M.$$

This shows that, if $u \sim v$ holds, we have $a_\ell^{-\tilde{\delta}_u} a_\ell^{\tilde{\delta}_v} \in M$. Suppose that, for $t \in \mathcal{T}$ and $x \in \mathcal{S}$, $xt \notin \mathcal{T}$ holds. Then there exists $u = \overline{xt} \in \mathcal{T}$ with $u \sim xt$. Write U for u^{-1} . Since $S \cap EK \trianglelefteq S$ holds, there exists $h \in S \cap EK \subseteq \ker(\gamma)$ so that $xt = hu$. This yields that $a_{\ell,t}^{\delta_x} = xt a_\ell TX = hu a_\ell U h^{-1} = h a_{\ell,u} h^{-1}$ and $a_{\ell,t}^{\delta_x \gamma} = a_{\ell,u}^\gamma = a_\ell^{\tilde{\delta}_U}$. Since $u \sim xt$ and $U \sim TX$ hold, we obtain

$$(a_{\ell,t}^{-1} a_{\ell,t}^{\gamma \chi})^{\delta_x \gamma} = a_{\ell,u}^{-\gamma} a_\ell^{\tilde{\delta}_T \tilde{\delta}_x} = a_\ell^{-\tilde{\delta}_U} a_\ell^{\tilde{\delta}_T \tilde{\delta}_x} \in M.$$

Thus $N \subseteq M$. It suffices to show that $M \subseteq N$ holds. Since M and N are both normal subgroups of $F(\mathcal{Z})$ and both are $\tilde{\Phi}^*$ -invariant, it suffices to prove that $a_\ell^{-\tilde{\delta}_{U_i}} a_\ell^{\tilde{\delta}_{V_i}} \in N = \ker(\psi)$ holds. Since $\tilde{\delta}_s \psi = \psi \tilde{\delta}_s$ and $\chi \gamma = \text{id}_{F(\mathcal{Z})}$ hold, we have that

$$\begin{aligned} (a_\ell^{-\tilde{\delta}_{U_i}} a_\ell^{\tilde{\delta}_{V_i}})^\psi &= a_\ell^{-\psi \tilde{\delta}_{U_i}} a_\ell^{\psi \tilde{\delta}_{V_i}} = a_\ell^{-\chi \gamma \psi \tilde{\delta}_{U_i}} a_\ell^{\chi \gamma \psi \tilde{\delta}_{V_i}} = a_{\ell,1}^{-\gamma \psi \tilde{\delta}_{U_i}} a_{\ell,1}^{\gamma \psi \tilde{\delta}_{V_i}} \\ &= a_{\ell,1}^{-\delta_{U_i} \gamma \psi} a_{\ell,1}^{\delta_{V_i} \gamma \psi} = (a_{\ell,1}^{-\delta_{U_i}} a_{\ell,1}^{\delta_{V_i}})^{\gamma \psi}. \end{aligned}$$

As $S \cap EK \trianglelefteq S$ and $\mathcal{T} \subseteq S$ hold, there exist $h \in S \cap EK = \langle s_{\ell,t} \mid 1 \leq \ell \leq n, t \in \mathcal{T} \rangle$ and $t = \overline{U_i^{-1}} \in \mathcal{T}$ with $U_i^{-1} = ht$. Thus $a_{\ell,1}^{\delta_{V_i}} = U_i^{-1} a_\ell U_i = h t a_\ell t^{-1} h^{-1} = h a_{\ell,t} h^{-1}$. Since $h \in \ker(\gamma)$ holds, we obtain $(a_{\ell,1}^{\delta_{V_i}})^\gamma = a_{\ell,t}^\gamma$. Since $U_i \sim V_i$ holds, we also have that $\overline{V_i^{-1}} = t$. Similarly, we obtain $(a_{\ell,1}^{\delta_{U_i}})^\gamma = a_{\ell,t}^\gamma$. Thus $a_{\ell,1}^{-\delta_{U_i}} a_{\ell,1}^{\delta_{V_i}} \in \ker(\gamma)$ and so $(a_\ell^{-\tilde{\delta}_{U_i}} a_\ell^{\tilde{\delta}_{V_i}})^\psi = 1$ or $a_\ell^{-\tilde{\delta}_{U_i}} a_\ell^{\tilde{\delta}_{V_i}} \in N$. Thus $M = N$. This shows that that factor group $F(\mathcal{Z})/N$ is invariantly finitely L -presented and so is our subgroup H . \square

Even if G/H is free, the finite L -presentation of $F(\mathcal{Z})/N$ in the proof of Theorem C contains the relations of a monoid presentation of the free group. It is not clear whether or not these relations can be omitted as was done in [4]. However, the result in [4] is a consequence of Theorem C even if these relations are not redundant:

Theorem 5.10 (Benli [4]) *Every finitely generated normal subgroup of a finitely presented group is invariantly finitely L -presented if the quotient is infinite cyclic.*

Proof. Since the quotient is free, the finitely presented group splits over its finitely generated normal subgroup and thus, by Theorem C, the subgroup is invariantly finitely L -presented. \square

Even if the finitely presented group does not split over its finitely generated subgroup, the subgroup is possibly invariantly finitely L -presented:

Theorem 5.11 *Every finitely generated normal subgroup of a finitely presented group is invariantly finitely L -presented if the quotient is free abelian with rank two.*

Proof. Let G be a finitely presented group and let $H \trianglelefteq G$ be finitely generated so that $G/H \cong \mathbb{Z} \times \mathbb{Z}$ holds. By Lemma 5.9, it suffices to construct a factor group $F(\mathcal{Z})/N$ which is invariantly finitely L -presented. Since $G/H \cong \mathbb{Z} \times \mathbb{Z}$ holds, there exists $t, u \in G$ so that $G/H = \langle tH, uH \rangle$ holds. Moreover, as H is finitely generated, there exist $a_1, \dots, a_m \in H$ so that $H = \langle a_1, \dots, a_m \rangle$ holds. Then $G = \langle a_1, \dots, a_m, t, u \rangle$ holds and there exists a finite set of relations \mathcal{R} with $G \cong \langle \{a_1, \dots, a_m, t, u\} \mid \mathcal{R} \rangle$. We choose as Schreier transversal $\mathcal{T} = \{t^i u^j \mid i, j \in \mathbb{Z}\}$. Then, by Lemma 5.3, the subgroup H is invariantly L -presented by $\langle \mathcal{Y} \mid \emptyset \mid \{\delta_u, \delta_U, \delta_t, \delta_T\} \mid \mathcal{R}^\tau \rangle$ where δ_x denotes the endomorphism of the free group $F(\mathcal{Y})$ that is induced by conjugation with $x \in \{u, U = u^{-1}, t, T = t^{-1}\}$, τ denotes the Reidemeister rewriting, and $\mathcal{Y} = \{a_{\ell, i, j}, t_{i, k} \mid i, j, k, l \in \mathbb{Z}, k \neq 0\}$ are the following Schreier generators:

$$\begin{aligned} a_{\ell, i, j} &= \gamma(t^i u^j, a_\ell) = t^i u^j a_\ell u^{-j} t^{-i}, \\ t_{i, j} &= \gamma(t^i u^j, t) = t^i u^j t u^{-j} t^{-1} t^{-i}, \\ u_{i, j} &= \gamma(t^i u^j, u) = t^i u^j u u^{-j} u^{-1} t^{-i}. \end{aligned}$$

Note that $t_{i, j} = 1$ if and only if $j = 0$ while $u_{i, j} = 1$ for each $i, j \in \mathbb{Z}$. The endomorphisms δ_t and δ_T are induced by the maps

$$\delta_t: \begin{cases} a_{\ell, i, j} \mapsto a_{\ell, i-1, j}, \\ t_{i, j} \mapsto t_{i-1, j}, \end{cases} \quad \text{and} \quad \delta_T: \begin{cases} a_{\ell, i, j} \mapsto a_{\ell, i+1, j}, \\ t_{i, j} \mapsto t_{i+1, j}, \end{cases}$$

for each $i, j \in \mathbb{Z}$; while δ_u and δ_U are induced by the maps

$$\delta_u: \begin{cases} a_{\ell, i, j} \mapsto (a_{\ell, i, j-1})^{t_{i-1, -1}^{-1} \cdots t_{0, -1}^{-1}}, & i \geq 0, j \in \mathbb{Z}, \\ a_{\ell, -i, j} \mapsto (a_{\ell, -i, j-1})^{t_{-i, -1} \cdots t_{-1, -1}}, & i \geq 0, j \in \mathbb{Z}, \\ t_{i, j} \mapsto (t_{i, j-1} t_{i, -1}^{-1})^{t_{i-1, -1}^{-1} \cdots t_{0, -1}^{-1}}, & i \geq 0, j \in \mathbb{Z}, \\ t_{-i, j} \mapsto (t_{-i, j-1} t_{-i, -1}^{-1})^{t_{-i, -1} \cdots t_{-1, -1}}, & i \geq 0, j \in \mathbb{Z}, \end{cases}$$

and

$$\delta_U: \begin{cases} a_{\ell, i, j} \mapsto (a_{\ell, i, j+1})^{t_{i-1, 1}^{-1} \cdots t_{0, 1}^{-1}}, & i \geq 0, j \in \mathbb{Z}, \\ a_{\ell, -i, j} \mapsto (a_{\ell, -i, j+1})^{t_{-i, 1} \cdots t_{-1, 1}}, & i \geq 0, j \in \mathbb{Z}, \\ t_{i, j} \mapsto (t_{i, j+1} t_{i, 1}^{-1})^{t_{i-1, 1}^{-1} \cdots t_{0, 1}^{-1}}, & i \geq 0, j \in \mathbb{Z}, \\ t_{-i, j} \mapsto (t_{-i, j+1} t_{-i, 1}^{-1})^{t_{-i, 1} \cdots t_{-1, 1}}, & i \geq 0, j \in \mathbb{Z}. \end{cases}$$

We will construct an invariant finite L -presentation for the subgroup H with generators $\mathcal{Z} = \{a_1, \dots, a_m\} \cup \{t_1\}$. Define an embedding $\chi: F(\mathcal{Z}) \rightarrow F(\mathcal{Y})$ that is induced by the map

$$\chi: \begin{cases} a_\ell \mapsto a_{\ell, 0, 0}, & \text{for each } 1 \leq \ell \leq m \\ t_1 \mapsto t_{0, 1}. \end{cases}$$

Write $\Phi = \{\delta_t, \delta_T, \delta_u, \delta_U\}$. For $y \in \mathcal{Z}$ and $\delta \in \Phi$, choose $y^{\chi\delta\gamma} \in F(\mathcal{Z})$ with

$$y^{-\chi\delta}(y^{\chi\delta\gamma})^\chi \in \ker(\pi). \quad (15)$$

Define $\iota: F(\mathcal{Z}) \rightarrow H$ by $\iota = \chi\pi$ where π denotes the free presentation $\pi: F(\mathcal{Y}) \rightarrow H$ that is given by H 's invariant L -presentation above. For each $\delta \in \Phi$, define an endomorphism $\tilde{\delta}: F(\mathcal{Z}) \rightarrow F(\mathcal{Z})$ that is induced by the map $y \mapsto y^{\chi\delta\gamma}$. Then, for each $\delta \in \Phi$ and $y \in \mathcal{Z}$, we obtain

$$y^{\hat{\iota}\delta} = y^{\chi\pi\hat{\delta}} = y^{\chi\delta\pi} = (y^{\chi\delta})^\pi = (y^{\chi\delta\gamma\chi})^\pi = (y^{\chi\delta\gamma})^{\chi\pi} = y^{\tilde{\delta}\iota}$$

and thus $\tilde{\delta}\iota = \gamma\hat{\delta}$. Write $\mathcal{X} = \{a_1, \dots, a_m, t, u\}$ and consider the following subgroups of the free group $F(\mathcal{X})$: Let $E = \langle a_1, \dots, a_m \rangle$ and $S = \langle t, u \rangle$ be finitely

generated subgroups of $F(\mathcal{X})$. Furthermore, let $K \trianglelefteq F(\mathcal{X})$ be the kernel of G 's free presentation. Then $G \cong F(\mathcal{X})/K$ and $H \cong EK/K$. Moreover, the normal subgroup $EK \trianglelefteq F(\mathcal{X})$ is supplemented by the finitely generated free group S ; i.e., $F(\mathcal{X}) = SEK$ holds. Thus $G/H \cong F(\mathcal{X})/EK \cong S/S \cap EK$. Since G/H is finitely presented, the free subgroup $S \cap EK$ is finitely generated as a normal subgroup. The Schreier generators \mathcal{Y} yield that the subgroups

$$EK = \langle \mathcal{Y} \rangle \quad \text{and} \quad S \cap EK = \langle t_{i,j} \mid i, j \in \mathbb{Z}, j \neq 0 \rangle$$

are freely generated. Moreover, we have that

$$\begin{aligned} S \cap EK &= \langle t_{i,j+1}t_{i,j}^{-1} \mid i, j \in \mathbb{Z} \rangle \\ &= \langle \dots t_{i,-2}t_{i,-3}^{-1}, t_{i,-1}t_{i,-2}^{-1}, t_{i,-1}^{-1}, t_{i,1}, t_{i,2}t_{i,1}^{-1}, t_{i,3}t_{i,2}^{-1}, \dots \mid i \in \mathbb{Z} \rangle. \end{aligned}$$

The latter subgroup is freely generated as the homomorphism ψ that is induced by the map

$$\psi: S \cap EK \rightarrow S \cap EK, \quad \begin{cases} t_{i,j} \mapsto t_{i,j+1}t_{i,j}^{-1}, & j < -1 \\ t_{i,-1} \mapsto t_{i,-1}^{-1}, \\ t_{i,1} \mapsto t_{i,1} \\ t_{i,j} \mapsto t_{i,j}t_{i,j-1}^{-1}, & j > 1 \end{cases}$$

is an automorphism of $S \cap EK$ whose inverse is induced by the map

$$\psi^{-1}: S \cap EK \rightarrow S \cap EK, \quad \begin{cases} t_{i,j} \mapsto t_{i,j}^{-1}t_{i,j+1}^{-1} \cdots t_{i,-1}^{-1}, & j < -1 \\ t_{i,-1} \mapsto t_{i,-1}^{-1}, \\ t_{i,1} \mapsto t_{i,1} \\ t_{i,j} \mapsto t_{i,j}t_{i,j-1} \cdots t_{i,1}, & j > 1. \end{cases}$$

Note that we have

$$t_{i,j+1}t_{i,j}^{-1} = t^i u^{j+1} t u^{-j-1} t^{-1} t^{-i} \cdot (t^i u^j t u^{-j} t^{-1} t^{-i})^{-1} = (t_{0,1})^{u^j t^i}.$$

In fact, every element in $S \cap EK$ has a unique representation as a word in the basis $\{t^i u^j \cdot t_{0,1} \cdot u^{-j} t^{-i} \mid i, j \in \mathbb{Z}\}$ where $t_{0,1} = utUT$ is a normal generator of $S \cap EK = \langle t_{0,1} \rangle^S$. More precisely, for $i \geq 0$ and $j > 0$, we have the following representatives in free subgroup $S \cap EK \leq F(\mathcal{Y})$:

$$\begin{aligned} t_{i,j} &= \left(t_{0,1}^{\delta_U^{j-1}} \cdot t_{0,1}^{\delta_U^{j-2}} \cdots t_{0,1} \right)^{\delta_T^i} & \text{and} & & t_{i,-j} &= \left(t_{0,1}^{-\delta_u^j} \cdot t_{0,1}^{-\delta_u^{j-1}} \cdots t_{0,1}^{-\delta_u} \right)^{\delta_T^i} \\ t_{-i,j} &= \left(t_{0,1}^{\delta_U^{-1}} \cdot t_{0,1}^{\delta_U^{-2}} \cdots t_{0,1} \right)^{\delta_T^i} & & & t_{-i,-j} &= \left(t_{0,1}^{-\delta_u^j} \cdot t_{0,1}^{-\delta_u^{j-1}} \cdots t_{0,1}^{-\delta_u} \right)^{\delta_T^i}. \end{aligned}$$

The Schreier generators $a_{\ell,i,j}$ are conjugates of the generators $a_{\ell,0,0}$ so that

$$\begin{aligned} a_{\ell,i,j} &= (a_{\ell,0,0})^{\delta_U^j \delta_T^i} & \text{and} & & a_{\ell,i,-j} &= (a_{\ell,0,0})^{\delta_u^j \delta_T^i} \\ a_{\ell,-i,j} &= (a_{\ell,0,0})^{\delta_U^j \delta_T^i} & & & a_{\ell,-i,-j} &= (a_{\ell,0,0})^{\delta_u^j \delta_T^i}. \end{aligned}$$

In particular, we can choose the following basis $\widehat{\mathcal{Y}}$ for the free subgroup EK :

$$\widehat{\mathcal{Y}} = \left\{ (a_{\ell,0,0})^{\delta_U^j \delta_T^i}, \dots, (a_{\ell,0,0})^{\delta_u^j \delta_T^i}, (t_{0,1})^{\delta_U^j \delta_T^i}, \dots, (t_{0,1})^{\delta_u^j \delta_T^i} \right\}_{i,j \geq 0}.$$

Define $\gamma: F(\widehat{\mathcal{Y}}) \rightarrow F(\mathcal{Z})$ to be induced by the map

$$\gamma: \begin{cases} (a_{\ell,0,0})^{\delta_U^j \delta_T^i} \mapsto (a_\ell)^{\tilde{\delta}_U^j \tilde{\delta}_T^i}, \\ (a_{\ell,0,0})^{\delta_U^j \delta_T^i} \mapsto (a_\ell)^{\tilde{\delta}_U^j \tilde{\delta}_T^i}, \\ (a_{\ell,0,0})^{\delta_u^j \delta_T^i} \mapsto (a_\ell)^{\tilde{\delta}_u^j \tilde{\delta}_T^i}, \\ (a_{\ell,0,0})^{\delta_u^j \delta_T^i} \mapsto (a_\ell)^{\tilde{\delta}_u^j \tilde{\delta}_T^i}, \end{cases} \quad \text{and} \quad \gamma: \begin{cases} (t_{0,1})^{\delta_U^j \delta_T^i} \mapsto (t_1)^{\tilde{\delta}_U^j \tilde{\delta}_T^i} \\ (t_{0,1})^{\delta_U^j \delta_T^i} \mapsto (t_1)^{\tilde{\delta}_U^j \tilde{\delta}_T^i} \\ (t_{0,1})^{\delta_u^j \delta_T^i} \mapsto (t_1)^{\tilde{\delta}_u^j \tilde{\delta}_T^i} \\ (t_{0,1})^{\delta_u^j \delta_T^i} \mapsto (t_1)^{\tilde{\delta}_u^j \tilde{\delta}_T^i}, \end{cases}$$

where $i, j \geq 0$. Then γ acts on the Schreier generators \mathcal{Y} as follows:

$$\gamma: \begin{cases} a_{\ell,i,j} \mapsto (a_\ell)^{\tilde{\delta}_U^j \tilde{\delta}_T^i}, \\ a_{\ell,-i,j} \mapsto (a_\ell)^{\tilde{\delta}_U^j \tilde{\delta}_t^i}, \\ a_{\ell,i,-j} \mapsto (a_\ell)^{\tilde{\delta}_u^j \tilde{\delta}_T^i}, \\ a_{\ell,-i,-j} \mapsto (a_\ell)^{\tilde{\delta}_u^j \tilde{\delta}_t^i}, \end{cases} \quad \text{and} \quad \gamma: \begin{cases} t_{i,j} \mapsto (t_1^{\tilde{\delta}_U^{j-1}} \cdots t_1)^{\tilde{\delta}_T^i}, \\ t_{-i,j} \mapsto (t_1^{\tilde{\delta}_U^{j-1}} \cdots t_1)^{\tilde{\delta}_t^i}, \\ t_{i,-j} \mapsto (t_1^{-\tilde{\delta}_u^j} \cdots t_1^{-\tilde{\delta}_u})^{\tilde{\delta}_T^i}, \\ t_{-i,-j} \mapsto (t_1^{-\tilde{\delta}_u^j} \cdots t_1^{-\tilde{\delta}_u})^{\tilde{\delta}_t^i}, \end{cases}$$

where $i \geq 0$ and $j > 0$. For $i \geq 0$ and $j > 0$, the element $t_{i,j} \in \mathcal{Y}$ is mapped by $\gamma\iota$ to

$$\begin{aligned} t_{i,j}^{\gamma\iota} &= (t_1^{\tilde{\delta}_U^{j-1}} \cdots t_1)^{\tilde{\delta}_T^i \iota} = (t_1^{\tilde{\delta}_U^{j-1} \tilde{\delta}_T^i} \cdots t_1^{\tilde{\delta}_T^i})^\iota = t_1^{\widehat{\delta}_U^{j-1} \widehat{\delta}_T^i} \cdots t_1^{\widehat{\delta}_T^i} \\ &= (t_1^{\chi^\pi \widehat{\delta}_U^{j-1} \tilde{\delta}_T^i} \cdots t_1^{\chi^\pi \tilde{\delta}_T^i}) = (t_{0,1}^{\tilde{\delta}_U^{j-1} \tilde{\delta}_T^i} \cdots t_{0,1}^{\tilde{\delta}_T^i})^\pi = (t_{i,j})^\pi \end{aligned}$$

because $\tilde{\delta}\iota = \iota\widehat{\delta}$ holds. Similarly, we obtain that $a_{\ell,i,j}^{\gamma\iota} = a_{\ell,i,j}^\pi$ holds. Thus $\gamma\iota = \pi$. Define the normal subgroup

$$N = \left\langle \bigcup_{\sigma \in \tilde{\Phi}^*} \left(\{ (y^{-1} y^{\gamma\chi})^{\delta\gamma} \}_{y \in \mathcal{Y} \setminus \mathcal{Z}, \delta \in \Phi} \right)^\sigma \right\rangle^{F(\mathcal{Z})}.$$

We prove that $F(\mathcal{Z})/N$ is invariantly finitely L -presented so that Lemma 5.9 applies. For $i \geq 0$ and $j > 0$, it holds that

$$\begin{aligned} (t_{i,j}^{-1} t_{i,j}^{\gamma\chi})^{\delta_T \gamma} &= t_{i+1,j}^{-\gamma} t_{i,j}^{\gamma \tilde{\delta}_T} = (t_1^{\tilde{\delta}_U^{j-1}} \cdots t_1)^{-\tilde{\delta}_T^{i+1}} (t_1^{\tilde{\delta}_U^{j-1}} \cdots t_1)^{\tilde{\delta}_T^i \tilde{\delta}_T} = 1, \\ (t_{-i,j}^{-1} t_{-i,j}^{\gamma\chi})^{\delta_t \gamma} &= t_{-i-1,j}^{-\gamma} t_{-i,j}^{\gamma \tilde{\delta}_t} = (t_1^{\tilde{\delta}_U^{j-1}} \cdots t_1)^{-\tilde{\delta}_t^{i+1}} (t_1^{\tilde{\delta}_U^{j-1}} \cdots t_1)^{\tilde{\delta}_t^i \tilde{\delta}_t} = 1, \\ (t_{i,-j}^{-1} t_{i,-j}^{\gamma\chi})^{\delta_T \gamma} &= t_{i+1,-j}^{-\gamma} t_{i,-j}^{\gamma \tilde{\delta}_T} = (t_1^{-\tilde{\delta}_u^j} \cdots t_1^{-\tilde{\delta}_u})^{-\tilde{\delta}_T^{i+1}} (t_1^{-\tilde{\delta}_u^j} \cdots t_1^{-\tilde{\delta}_u})^{\tilde{\delta}_T^i \tilde{\delta}_T} = 1, \\ (t_{-i,-j}^{-1} t_{-i,-j}^{\gamma\chi})^{\delta_t \gamma} &= t_{-i-1,-j}^{-\gamma} t_{-i,-j}^{\gamma \tilde{\delta}_t} = (t_1^{-\tilde{\delta}_u^j} \cdots t_1^{-\tilde{\delta}_u})^{-\tilde{\delta}_t^{i+1}} (t_1^{-\tilde{\delta}_u^j} \cdots t_1^{-\tilde{\delta}_u})^{\tilde{\delta}_t^i \tilde{\delta}_t} = 1. \end{aligned}$$

For $i = 0$, we also have that

$$\begin{aligned} (t_{0,j}^{-1} t_{0,j}^{\gamma\chi})^{\delta_U \gamma} &= (t_{0,j+1} t_{0,1}^{-1})^{-\gamma} t_{0,j}^{\gamma \tilde{\delta}_U} = (t_1^{\tilde{\delta}_U^j} \cdots t_1^{\tilde{\delta}_U} \cdot t_1 \cdot t_1^{-1})^{-1} (t_1^{\tilde{\delta}_U^{j-1}} \cdots t_1)^{\tilde{\delta}_U} = 1, \\ (t_{0,-j}^{-1} t_{0,-j}^{\gamma\chi})^{\delta_u \gamma} &= (t_{0,-j-1} t_{0,-1}^{-1})^{-\gamma} t_{0,-j}^{\gamma \tilde{\delta}_u} = (t_1^{-\tilde{\delta}_u^{j+1}} \cdots t_1^{-\tilde{\delta}_u} \cdot t_1^{\tilde{\delta}_u})^{-1} (t_1^{-\tilde{\delta}_u^j} \cdots t_1^{\tilde{\delta}_u})^{\tilde{\delta}_u} = 1. \end{aligned}$$

However, we also need to consider the image $t_{i,j}^{\delta_U} = t_{0,1}^{\delta_T^i \delta_U}$ with $i > 0$. Notice that in the finitely presented monoid $S/S \cap EK$ the following holds:

$$\begin{aligned} TU &= UT \cdot tuTU &= UT \cdot (utUT)^{-1} &= UT \cdot t_{0,1}^{-1}, \\ Tu &= uT \cdot tUTu &= uT \cdot (utUT)^{\delta_u} &= uT \cdot t_{0,1}^{\delta_u}, \\ tU &= U \cdot (utUT) \cdot t &= U \cdot t_{0,1} \cdot t &= Ut \cdot t_{0,1}^{\delta_t}, \\ tu &= u \cdot (UtuT) \cdot t &= u \cdot t_{0,-1} \cdot t &= ut \cdot t_{0,1}^{-\delta_u \delta_t}. \end{aligned}$$

Denote by $\Delta(x): F(\mathcal{Y}) \rightarrow F(\mathcal{Y})$, $g \mapsto x^{-1}gx$ the inner automorphism of $F(\mathcal{Y})$ that is induced by conjugation with $x \in F(\mathcal{Y})$. Then $\delta \in \Phi = \{\delta_u, \delta_U, \delta_t, \delta_T\}$ satisfy

$$\begin{aligned} \delta_T \delta_U &= \delta_U \delta_T \cdot \Delta(t_{0,1}^{-1}), & \delta_t \delta_U &= \delta_U \delta_t \cdot \Delta(t_{0,1}^{\delta_t}), \\ \delta_T \delta_u &= \delta_u \delta_T \cdot \Delta(t_{0,1}^{\delta_u}), & \delta_t \delta_u &= \delta_u \delta_t \cdot \Delta(t_{0,1}^{-\delta_u \delta_t}). \end{aligned}$$

We prove that $F(\mathcal{Z})/N$ is invariantly finitely L -presented by $\langle \{a_1, \dots, a_m, t_1\} \mid \emptyset \mid \tilde{\Phi} \mid \mathcal{V} \rangle$ where the iterated relations in \mathcal{V} are given by

$$\mathcal{V} = \left\{ y^{-1} y^{\tilde{\delta}_t \tilde{\delta}_T}, \dots, y^{-1} y^{\tilde{\delta}_U \tilde{\delta}_u}, y^{-\tilde{\delta}_T \tilde{\delta}_U} y^{\tilde{\delta}_U \tilde{\delta}_T \Delta(t_1^{-1})}, \dots, y^{-\tilde{\delta}_t \tilde{\delta}_u} y^{\tilde{\delta}_u \tilde{\delta}_t \Delta(t_1^{-\delta_u \tilde{\delta}_t})} \right\}_{y \in \mathcal{Z}}$$

that is, we prove that $M = \langle \bigcup_{\bar{\sigma} \in \bar{\Phi}} \mathcal{V}^{\bar{\sigma}} \rangle^{F(\mathcal{Z})}$ and N coincide. We first note that

$$\begin{aligned} N \ni (t_{1,1}^{-1} t_{1,1}^{\gamma\chi})^{\delta_U\gamma} &= t_{1,1}^{-\delta_U\gamma} t_{1,1}^{\tilde{\delta}_U} = t_{0,1}^{-\delta_T\delta_U\gamma} t_1^{\tilde{\delta}_T\tilde{\delta}_U} = t_{0,1}^{-\delta_U\delta_T\cdot\Delta(t_{0,1}^{-1})\gamma} t_1^{\tilde{\delta}_T\tilde{\delta}_U} \\ &= t_{0,1}^{-\delta_U\delta_T\gamma\cdot\Delta(t_1^{-1})} t_1^{\tilde{\delta}_T\tilde{\delta}_U} = t_1^{-\tilde{\delta}_U\tilde{\delta}_T\cdot\Delta(t_1^{-1})} t_1^{\tilde{\delta}_T\tilde{\delta}_U} \in \mathcal{V}. \end{aligned}$$

Similar computations show that the elements in \mathcal{V} appear among the normal generators of N . Thus $M \subseteq N$. On the other hand, for $i > 0$ and $j > 0$, we have that

$$\begin{aligned} t_{i,j}^{\delta_U\gamma} &= (t_{0,1}^{\delta_U^{j-1}} \cdots t_{0,1})^{\delta_U^i\delta_U\gamma} \\ &= (t_{0,1}^{\delta_U^{j-1}} \cdots t_{0,1})^{\delta_U^{i-1}\delta_U\delta_T\cdot\Delta(t_{0,1}^{-1})\gamma} \\ &= \cdots = (t_{0,1}^{\delta_U^{j-1}} \cdots t_{0,1})^{\delta_U\delta_T^i\cdot\Delta(t_{0,1}^{-\delta_T^{i-1}})\Delta(t_{0,1}^{-\delta_T^{i-2}})\cdots\Delta(t_{0,1}^{-1})\gamma} \\ &= (t_{0,1}^{\delta_U^{j-1}} \cdots t_{0,1})^{\delta_U\delta_T^i\gamma\cdot\Delta(t_1^{-\delta_T^{i-1}})\Delta(t_1^{-\delta_T^{i-2}})\cdots\Delta(t_1^{-1})} \\ &= (t_{0,1}^{\delta_U^j\delta_T^i\gamma} \cdots t_{0,1}^{\delta_U\delta_T^i\gamma})\Delta(t_1^{-\delta_T^{i-1}})\Delta(t_1^{-\delta_T^{i-2}})\cdots\Delta(t_1^{-1}) \\ &= (t_1^{\delta_U^{j-1}} \cdots t_1)^{\tilde{\delta}_U\tilde{\delta}_T^i\cdot\Delta(t_1^{-\delta_T^{i-1}})\Delta(t_1^{-\delta_T^{i-2}})\cdots\Delta(t_1^{-1})} \\ &= (t_1^{\delta_U^{j-1}} \cdots t_1)^{\tilde{\delta}_U\tilde{\delta}_T\Delta(t_1^{-1})\tilde{\delta}_T^{i-1}\cdot\Delta(t_1^{-\delta_T^{i-2}})\cdots\Delta(t_1^{-1})} \\ &\equiv (t_1^{\delta_U^{j-1}} \cdots t_1)^{\tilde{\delta}_T\tilde{\delta}_U\tilde{\delta}_T^{i-1}\cdot\Delta(t_1^{-\delta_T^{i-2}})\cdots\Delta(t_1^{-1})} \pmod{M} \\ &\equiv \cdots \equiv (t_1^{\delta_U^{j-1}} \cdots t_1)^{\tilde{\delta}_T^i\tilde{\delta}_U} \pmod{M} \end{aligned}$$

and $(t_1^{\delta_U^{j-1}} \cdots t_1)^{\tilde{\delta}_T^i\tilde{\delta}_U} = (t_{i,j})^{\gamma\tilde{\delta}_U}$. Thus $(t_{i,j}^{-1} t_{i,j}^{\gamma\chi})^{\delta_U\gamma} \in M$. It follows analogously for the other normal generators of N that these are contained in M . Thus $F(\mathcal{Z})/N$ is invariantly finitely L -presented and so is our subgroup H . \square

By [13, Theorem 6.1], every finite index subgroup H of an invariantly finitely L -presented group $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$ is invariantly finitely L -presented whenever the substitutions $\sigma \in \Phi$ induce endomorphisms of the subgroup H . This allows us to prove Theorem D using the results in Theorem 5.11 and Theorem 5.10:

Proof of Theorem D. Let G be a finitely presented group and let $H \trianglelefteq G$ be a finitely generated normal subgroup so that G/H is abelian with torsion-free rank at most two. Since G is finitely generated, G/H is a finitely generated abelian group and so it decomposes into $G/H \cong \mathbb{Z}^\ell \times T$ with torsion subgroup T and torsion-free rank $\ell \leq 2$. Denote by $U \leq G$ the full preimage of the torsion subgroup T in G . Then $G/U \cong \mathbb{Z}^\ell$ and $[U : H] < \infty$ hold. If $\ell = 0$ holds, H has finite index in G and thus it is invariantly finitely L -presented by [13, Theorem 6.1]. If either $\ell = 1$ or $\ell = 2$ holds, the subgroup $U \trianglelefteq G$ is invariantly finitely L -presented by Theorem 5.10 or Theorem 5.11. Each substitution in the L -presentation of U is induced by conjugation within the finitely presented group G . Since H is a normal subgroup of G each substitution of the finite L -presentation of U stabilizes the subgroup H . Thus [13, Theorem 6.1] applies to the finite index subgroup $H \trianglelefteq U$ and it shows that H is invariantly finitely L -presented. \square

In the proof of Theorem 5.11, it is essential that the elements $g \in S \cap EK$ have a unique representation in the basis $\{t^i s^j \cdot t_{0,1} \cdot s^{-j} t^{-i} \mid i, j \in \mathbb{Z}\}$. This allows us to define the epimorphism $\gamma : F(\mathcal{V}) \rightarrow F(\mathcal{Z})$ so that it maps conjugates by elements of the Schreier transversal to images of automorphisms which are induced by conjugation with a Schreier transversal. Since $S/S \cap EK$ is finitely presented, we can always choose finitely many Schreier generators $\mathcal{W} \subseteq \mathcal{V}$ so that $S \cap EK$ is generated,

as a normal subgroup, by \mathcal{W} . In our proof of Theorem 5.11 the conjugates of these elements in \mathcal{W} by elements of the Schreier transversal form a basis for the subgroup $S \cap EK$. This is no longer possible for $G/H \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$:

Remark 5.12 Consider the notation from the proof of Theorem 5.11. For $G/H \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, we choose as Schreier transversal $\mathcal{T} = \{r^i s^j t^k \mid i, j, k \in \mathbb{Z}\}$ and we obtain the Schreier generators:

$$\begin{aligned} a_{\ell,i,j,k} &= \gamma(r^i s^j t^k, a_\ell) &= r^i s^j t^k a_\ell t^{-k} s^{-j} r^{-i}, \\ s_{i,j,k} &= \gamma(r^i s^j t^k, s) &= r^i s^j (t^k s t^{-k} s^{-1}) s^{-j} r^{-i}, \\ r_{i,j,k} &= \gamma(r^i s^j t^k, r) &= r^i (s^j t^k r t^{-k} s^{-j} r^{-1}) r^{-i}, \\ t_{i,j,k} &= \gamma(r^i s^j t^k, t) &= 1, \end{aligned}$$

where $s_{i,j,k} = 1$ if and only if $k = 0$ while $r_{i,j,k} = 1$ if and only if $(j, k) = (0, 0)$. Then

$$EK = \langle a_{\ell,i,j,k}, s_{i,j,o}, r_{i,p,q} \mid 1 \leq \ell \leq m, i, j, k, o, p, q \in \mathbb{Z}, o \neq 0, (p, q) \neq (0, 0) \rangle$$

is freely generated and so is

$$S \cap EK = \langle s_{i,j,o}, r_{i,p,q} \mid i, j, o, p, q \in \mathbb{Z}, o \neq 0, (p, q) \neq (0, 0) \rangle.$$

Since $G/H \cong S/S \cap EK \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is finitely presented, the subgroup $S \cap EK$ is finitely generated as a normal subgroup of S . In particular, we have that

$$S/S \cap EK = \langle r, s, t \mid \underbrace{tst^{-1}s^{-1}}_{=s_{0,0,1}}, \underbrace{trt^{-1}r^{-1}}_{=r_{0,0,1}}, \underbrace{sr s^{-1}r^{-1}}_{=r_{0,1,0}} \rangle$$

so that $S \cap EK = \langle s_{0,0,1}, r_{0,0,1}, r_{0,1,0} \rangle^S$ holds. The normal generators of $S \cap EK$ satisfy

$$\begin{aligned} r^i s^j t^k \cdot s_{0,0,1} \cdot t^{-k} s^{-j} r^{-i} &= s_{i,j,k+1} \cdot s_{i,j,k}^{-1}, \\ r^i s^j t^k \cdot r_{0,0,1} \cdot t^{-k} s^{-j} r^{-i} &= r_{i,j,k+1} \cdot r_{i,j,k}^{-1}, \\ r^i s^j t^k \cdot r_{0,1,0} \cdot t^{-k} s^{-j} r^{-i} &= s_{i,j,k} \cdot r_{i,j+1,k} \cdot s_{i+1,j,k}^{-1} \cdot r_{i,j,k}^{-1}. \end{aligned}$$

It can be seen easily (e.g. using GAP) that

$$U = \{s_{i,j,k+1} s_{i,j,k}^{-1}, r_{i,j,k+1} r_{i,j,k}^{-1}, s_{i,j,k} r_{i,j+1,k} s_{i+1,j,k}^{-1} r_{i,j,k}^{-1}\}_{i,j,k \in \mathbb{Z}}$$

is not a basis for $S \cap EK$. Therefore the ideas in the proof of Theorem 5.11 do not apply.

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References

- [1] L. Bartholdi. Endomorphic presentations of branch groups. *J. Algebra*, 268(2):419–443, 2003.
- [2] L. Bartholdi, B. Eick, and R. Hartung. A nilpotent quotient algorithm for certain infinitely presented groups and its applications. *Internat. J. Algebra Comput.*, 18(8):1321–1344, 2008.

- [3] L. Bartholdi and O. Siegenthaler. The twisted twin of the Grigorchuk group. *Internat. J. Algebra Comput.*, 24(4):465–488, 2010.
- [4] M. Benli. Indicible groups and endomorphic presentations. *Glasgow Math. J.*, 2011. Available on CJO 2011 doi:10.1017/S0017089511000632.
- [5] P. de la Harpe. *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.
- [6] R. I. Grigorchuk. On Burnside’s problem on periodic groups. *Funktsional. Anal. i Prilozhen.*, 14(1):53–54, 1980.
- [7] R. I. Grigorchuk. An example of a finitely presented amenable group that does not belong to the class EG. *Mat. Sb.*, 189(1):79–100, 1998.
- [8] R. I. Grigorchuk. On the system of defining relations and the Schur multiplier of periodic groups generated by finite automata. In *Groups St. Andrews 1997 in Bath, I*, volume 260 of *London Math. Soc. Lecture Note Ser.*, pages 290–317. Cambridge Univ. Press, Cambridge, 1999.
- [9] N. Gupta and S. Sidki. On the Burnside problem for periodic groups. *Math. Z.*, 182(3):385–388, 1983.
- [10] R. Hartung. Approximating the Schur multiplier of certain infinitely presented groups via nilpotent quotients. *LMS J. Comput. Math.*, 13:260–271, 2010. Preprint available as arxiv.org/abs/1106.1098.
- [11] R. Hartung. Coset enumeration for certain infinitely presented groups. *Internat. J. Algebra Comput.*, 21(8):1369–1380, 2011. Preprint available as arxiv.org/abs/1106.0259.
- [12] R. Hartung. Investigating self-similar groups using their finite L -presentation. Preprint available as arxiv.org/abs/1204.4279, 2012.
- [13] R. Hartung. A Reidemeister-Schreier theorem for finitely L -presented groups. *Internat. Elec. J. Algebra*, 11:125–159, 2012. Preprint available as arxiv.org/abs/1108.2403.
- [14] G. Higman. Subgroups of finitely presented groups. *Proc. Roy. Soc. Ser. A*, 262:455–475, 1961.
- [15] W. Krull. Über verallgemeinerte endliche Abelsche Gruppen. *M. Z.*, 23:161–196, 1925.
- [16] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory*. Springer-Verlag, Berlin, 1977. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.
- [17] I. G. Lysënok. A set of defining relations for the Grigorchuk group. *Mat. Zametki*, 38(4):503–516, 634, 1985.
- [18] E. Noether. Hyperkomplexe Größen und Darstellungstheorie. *M. Z.*, 30:641–692, 1929.
- [19] S. Sidki. On a 2-generated infinite 3-group: the presentation problem. *J. Algebra*, 110(1):13–23, 1987.
- [20] H. Tietze. Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten. *Monatsh. Math. Phys.*, 19(1):1–118, 1908.

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